To compose a number is to compose a cardinality: evidence from intensional contexts

Luis Alonso-Ovalle
McGill University
Montréal, Québec, Canada
luis.alonso-ovalle@mcgill.ca

Bernhard Schwarz
McGill University
Montréal, Québec, Canada
bernhard.schwarz@mcgill.ca

Abstract

DPs of the form *the number of NP* can combine with predicates of numbers or with predicates of ordinary entities. The interpretation of such DPs in intensional contexts favours an account of this flexibility based on syntactic ambiguity (Alonso-Ovalle and Schwarz 2024) over an account based on constructing numbers from ordinary entities (cf. Scontras 2017).

1 Introduction

DPs whose NP is headed by the quantity noun *number* (NDPs), like (1), can saturate predicates of numbers, like *is below ten* in (2-a), and also predicates of ordinary entities, like *hire* in (2-b). This flexibility is surprising. Number predication is expected if NDPs are headed by the quantity noun, as in the parse in (3): if *number* is a predicate of numbers, NDPs should denote a number. But then, if NDPs denote numbers, how does entity predication arise?

(1) the number of cooks Al hired

(2) a. [The number of cooks Al hired] is below ten.
b. Bo hired [the number of cooks Al hired].

(3) [$_{\text{DP}}$ the [$_{\text{NP}}$ [$_{\text{N}}$ **number**] of cooks Al hired]]

In what we call the "rich number" approach, pioneered for closely related data in Scontras 2017, numbers are not primitives like cardinalities, but are constructed from ordinary entities. The simplest implementation of this idea takes numbers to be *sets* of pluralities of ordinary entities that have the same cardinality. In a world where Al hired three cooks, for example, the number denoted by (3) is then the set containing all cook pluralities of cardinality 3. These are the pluralities accessed in entity predication: (2-b) says that Bo hired at least one of them.[1]

In the "rich syntax" approach entertained in Alonso-Ovalle and Schwarz 2024, syntax enables entity predication. NDPs are taken to be structurally ambiguous. The parse in (3) is just one parse of (1). A second parse, in (4), contains two occurrences of *cooks*—one unpronounced. The higher occurrence of *cooks* is the head of the main NP, and is modified by the DP in (3).

(4) [$_{\text{DP}}$ [$_{\text{NP}}$ [[$_{\text{DP}}$ the [$_{\text{NP}}$ [$_{\text{N}}$ number] of cooks Al hired]]] [$_{\text{N}}$ **cooks**]]]

The DP structure (3) is taken to denote a cardinality, such as 3 in a world where Al hired three cooks. Hence (3) supports number predication. The parse in (4) yields entity predication. At the level of the main NP, (4) parallels cases of modification by a numeral, as in *three cooks*.

[1] Reconstructing numbers in terms of entities is an old idea. Frege and Russell treat cardinal numbers as sets of sets with the same cardinality (see e.g. Hatcher 1990; Dummett 1991). Semanticists proposed, more generally, reconstructing degrees as equivalence classes of ordinary entities (see e.g. Cresswell 1976, Klein 1980 or Bale 2006.) Scontras 2017 does not construe rich numbers (more generally, amounts) as sets of entities, but as *nominalized properties of entities.* We stick to sets for simplicity of exposition, without affecting the argument.

Just like composing the numeral modifier with the head *cooks* in *three cooks* yields the set of cook pluralities of cardinality 3, composing the DP modifier with *cooks* in (4) yields the set of cook pluralities whose cardinality is the number of cooks Al hired. In a world where Al hired three cooks, this set is again the set of cook pluralities of cardinality 3. And again, (2-b) says that Bo hired at least one of them. Notably, this view makes it possible to maintain that numbers are cardinalities, understood as primitives, and that *number* is a predicate of cardinalities.

This paper presents a new argument for the rich syntax approach. The argument references NDPs in intensional contexts, where entity predication allows for "number transparency", truth conditions that reference a cardinality determined in the actual world. For instance, (5) can convey that Bo wants to hire a set of cooks whose cardinality is the number of cooks that Al actually hired. Since this cardinality can be given by the denotation of the modifier DP in (4), the rich syntax approach renders number transparency unsurprising. We argue that, in contrast, the rich number approach fails to offer a compositional path to number transparency.

(5) Bo wants [PRO to hire [the number of cooks Al hired]].

2 Entity predication

The sentence in (2-b) has truth conditions that make an existential statement (cf. Scontras 2017): they convey that Bo hired *some* plurality of cooks whose cardinality is the cardinality of the set of cooks that Al hired. We can state those truth conditions as in (6). This statement makes reference to the set in (7), whose description, in turn, refers to the cardinality in (8). The cardinality in (8) is the (maximal) number of cooks that Al hired. The set in (7) is the set of cook pluralities of that cardinality. The truth conditions in (6) quantify existentially over that set: they convey that Bo hired one of the pluralities in (7). In a world where Al hired three cooks, the cardinality in (8) is 3, the set in (7) is the set that contains all pluralities of three cooks, and the condition for the truth of (2-b) is that Bo hired at least one of them.

(6) $\exists x \in (7)[\text{hired(Bo)(x)}]$

(7) *Domain component*: $\{y|\ \text{cooks(y)} \wedge |y| = (8)\}$

(8) *Cardinality component*: $\max \{n|\ \exists x \in \{z|\ \text{cooks(z)} \wedge |z| = n\}[\text{hired(Al)(x)}]\}$

We will see next that both the rich syntax approach and the rich number approach derive these truth conditions, but that the approaches differ in terms of which syntactic constituents, if any, contribute the domain component in (7) and the cardinality component in (8). This difference will become consequential once we attend to NDPs in intensional contexts.

2.1 Composition: rich syntax

One of the parses that the rich syntax approach assumes for the NDP in (1) is repeated in (9). Here the noun *number* is assumed to be a predicate of cardinalities, and so the whole definite DP is assumed to denote one: the cardinality of the set of cooks that Al hired, in (8). Since (9) denotes a cardinality, it can saturate number predicates like *is below ten.*

(9) $[_{\text{DP}}$ the $[_{\text{NP}}$ $[_{\text{N}}$ **number**] of cooks Al hired]]

The second parse is repeated in (10). This is the parse involved in cases of entity predication like (2-b), repeated in (11), with the truth conditions given in (6) to (8).

(10) $[_{\text{DP}}$ $[_{\text{NP}}$ $\boxed{[_{\text{DP}} \text{ the } [_{\text{NP}}\ [_{\text{N}} \text{ number }] \text{ of cooks Al hired }]\ ]}$ $[_{\text{N}}$ **cooks**]]]

(11) Bo hired [the number of cooks Al hired].

In (10), the main NP contributes the domain component in (7), and the modifying DP the cardinality component in (8). Modification restricts the denotation of the head noun, the set of all cook pluralities, to (7), the set of all cook pluralities with the cardinality in (8). Note the parallel to the NP *three cooks*, where *three* can be taken to restrict the denotation of *cooks* to the set of cook pluralities with cardinality 3. This restricting function could be enabled by a covert predicate MANY, as in *[$_{AP}$ three MANY] cooks* and in (12) (cf. Hackl 2000), which we will nevertheless omit from LFs. As for the source of the existential quantification in (6), we again merely assume that it is parallel to cases with numerals, like *Bo hired three cooks*. The quantification could be sourced to a covert existential head of the main DP. In an alternative composition, which we will adopt for ease of exposition, existence quantification is attached to the entity predicate, given a predication postulate such as (13).[2,3]

(12) $[_{DP}$ $[_{NP}$ $[_{AP}$ $[_{DP}$ the $[_{NP}$ $[_{N}$ number] of cooks Al hired]] MANY] $[_{N}$ **cooks**]]]

(13) $[\![\text{hired}]\!](A)(y) \Leftrightarrow \exists x[x \in A \wedge [\![\text{hired}]\!](x)(y)]$

2.2 Composition: rich numbers

In an alternative based on Scontras 2017 (cf. Moltmann 2013), NDPs are unambiguously parsed as definite DPs, as in (14), which again repeats (3). Crucially, this definite DP is *not* taken to denote the cardinality (8), repeated in (15), but a "rich number": the set of equinumerous cook pluralities in (7), the domain component, repeated in (16). Under this analysis, it is the main DP, not the NP, that contributes the domain component.

(14) $[_{DP}$ the $[_{NP}$ $[_{N}$ **number**] of cooks Al hired]]

(15) $\max \{n | \exists x \in \{z | \text{cooks}(z) \wedge |z| = n\}[\text{hired}(Al)(x)]\}$

(16) $\{y | \text{cooks}(y) \wedge |y| = (15)\}$

Rich numbers convey information about both entities and cardinalities—(16) is a set of entities that are cooks and that share a certain cardinality. We then expect both quantity and entity predication to be possible. Entity predication simply conveys that the predicate is true of at least one member of the rich number, as in (17-b) for (17-a)—as before, the source of the existential quantification could be the predicate postulate in (13). Number predication can be taken to make a claim about the cardinality of the pluralities in a rich number, as in (18).

(17) a. Bo hired [the number of cooks Al hired].
b. $\exists x \in (16)[\text{hired}(Bo)(x)]$

(18) $[\![\text{is below ten}]\!](A) \Leftrightarrow \exists x[x \in A \wedge |x| < 10]$

The proposal in Scontras 2017 also suggests an internal composition for the DP. Within the

[2] In Scontras 2017, equivalences like (13) are the result of "derived kind predication", a composition principle adapted from Chierchia 1998, who builds on Carlson 1977. See Alonso-Ovalle and Schwarz 2024 for a critique.

[3] For the internal composition of (9), we assume partial reconstruction of the NP head of the relative clause, as in (i). Within the relative clause, the object DP may be headed by a covert existential determiner, or composition may proceed via (13). Either way, the relative clause denotes a set of numbers (ii-a), like *number* does (ii-b). The head noun and the relative clause can then compose intersectively. *The* picks up the largest number in the denotation of the main NP. The LF (i) is reminiscent of analyses of *how many NP* questions that decompose the wh-phrase into a wide scope quantifier ranging over cardinalities and a piece equivalent to the relative clause internal object DP in (i), which can be reconstructed. See Cresti 1995 and Rullmann 1995 and references therein.

(i) $[_{DP}$ the $[_{NP}$ $[_{NP}$ number] $[_{CP}$ wh_n Al hired $[_{DP}$ n MANY (of) cooks]]]

(ii) a. $[\![[_{CP}\ wh_n$ [Al hired $[_{DP}$ n MANY $\underline{\text{(of) cooks}}$]]] $]\!] = \{n \mid \exists y[\text{cooks}(y) \wedge |y| = n \wedge \text{hired}(Al)(y)]\}$
b. $[\![\text{number}]\!] = \{0,1,2,3 \ldots\}$

main NP, the relative clause *Al hired* can be parsed as modifying the NP *number of cooks*, as detailed in (19). The NP *number of cooks* will denote the set of all rich numbers made of cooks, in (20-a). Each such rich number is a set that contains all cook pluralities of a given cardinality. The relative clause can likewise denote a set of rich numbers: those that contain a plurality that Al hired, (20-b). The NP *number of cooks* and the relative clause combine intersectively. That yields the set that contains all rich numbers made of cooks that contain at least one plurality that Al hired. If Al hired exactly three cooks, the highest NP then denotes a set that contains the set of all atomic cooks, the set of all two cook pluralities, and the set of all three cook pluralities. *The* picks up the last set, on the basis that it contains the largest pluralities.

(19) $[_{NP}$ $[_{NP}$ $[_{N}$ **number**] of cooks] $[_{CP}$ wh Al hired t]]

(20) a. {N | N is a rich number ∧ ∀x ∈ N [cooks(x)]}
b. {N | N is a rich number ∧ ∃x ∈ N [hired(Al)(x)]}

A central feature of the rich number analysis of NDPs is that, unlike the rich syntax analysis, it does not posit any constituent that denotes a cardinality. In fact, we can understand the rich number analysis as the claim that there is no such constituent in NDPs, and perhaps more broadly in language.

3 Number transparency

We now consider cases of entity predication with NDPs in an intensional context, as in (5), repeated in (21). This sentence is ambiguous between readings that we call "opaque" and "number transparent". In the opaque reading, (21) reports on the relation desired by Bo between the numbers of cooks hired by Al and Bo. It conveys that Bo wants to match Al in terms of the number of cooks hired, that Bo wishes to do no worse than Al on that scale.

(21) Bo wants [PRO to hire [the number of cooks Al hired]].

We can also state this reading as in (22), with reference to the domain component in (23) and the cardinality component in (24), where, crucially, the meta-language predicates *cooks* and *hire* are applied to desire worlds, represented by *w*, as opposed to the actual world. Given this, (22) states that for each world w in the set of Bo's desire worlds, Acc, Bo hires at least m cooks in w, where m is the (maximal) number of entities that are cooks in w and that Al hires in w.

(22) $\text{Acc} \subseteq \{w \mid \exists x \in (23)[\text{hired}(w)(\text{Bo})(x)]\}$

(23) *Domain component:* $\{y \mid \text{cooks}(w)(y) \wedge |y| = (24)\}$

(24) *Cardinality component:* $\max \{n \mid \exists x \in \{z \mid \text{cooks}(w)(z) \wedge |z| = n\}[\text{hired}(w)(\text{Al})(x)]\}$

The "number transparent" reading of (21), in contrast, describes a desire of Bo that imposes a lower bound on the number of cooks Bo hires in absolute terms, not in comparison to Al. It states that Bo wants to hire some plurality of cooks whose cardinality is the number of cooks that Al happens to actually have hired. So under the assumption that the number of cooks that Al hired is three, (21) on that reading is equivalent to (25).

(25) Bo wants to hire three cooks.

The number transparent reading can also be stated as in (26), with reference to the domain component in (27) and the cardinality component in (28). In the cardinality term in (28), the meta-language predicates *cooks* and *hire* are now both evaluated in the actual world @. The truth conditions in (26) state that for every desire world w, Bo hires at least m cooks in w, where m is the (maximal) number of cooks in the actual world that Al hired in the actual world.

(26) $Acc \subseteq \{w|\ \exists x \in (27)[hired(w)(Bo)(x)]\}$

(27) *Domain component:* $\{y|\ cooks(w)(y) \wedge |y| = (28)\}$

(28) *Cardinality component:* $\max \{n|\ \exists x \in \{z|\ cooks(@)(z) \wedge |z| = n\}[hired(@)(Al)(x)]\}$

The opaque and number transparent interpretation only differ in how the cardinality component is arrived at, in particular, in what worlds the metalanguage predicates *cooks* and *hired* take as arguments: either the desire worlds or the actual world. With this in mind, let us now discuss how the rich syntax analysis derives the opaque and number transparency readings.

3.1 Number transparency: rich syntax

On the rich syntax account, number transparent reading are unsurprising and compatible with various views of how transparent readings arise in general. For concreteness only, we will focus here on one familiar option. In this approach, the world arguments of predicates are syntactically represented, and a predicate's transparent interpretation can result from the assumption that its world argument is filled by a covert world pronoun @ that picks out the actual world (Percus 2000). Under this view, the embedded clause in (21) can be assigned the LF in (29).

(29) λw PRO to hire$_w$ [$_{DP}$ [$_{NP}$ [$_{DP}$ the number of cooks$_@$ Al hired$_@$]] [$_N$ cooks$_w$]]]

We have seen that the modifying DP in (29) contributes to the semantic composition the cardinality component of the truth conditions that arise from entity predication. In (29), the occurrences of *cook* and *hired* in this DP are indexed with the world pronoun @ denoting the actual world. This yields the cardinality component in (28) and, in consequence, the number transparent reading of (21): with its embedded clause parsed as in (29), (21) conveys that Bo wants to hire some plurality of cooks that has the cardinality of the plurality of actual cooks that Al actually hired. So in a world where that cardinality is 3, (21) in the parse (29) conveys that Al wants to hire (at least) three cooks.

In contrast, in the parse in (30) all the word pronouns within the modifying DP are bound within the embedded clause. This yields the target domain cardinality in (27), and so (30) characterizes the set of worlds w such that Bo hires a cook plurality in w whose size is the (maximal) number of entities that are cooks in w and that Al hired in w. The intended opaque reading of (21) then falls out compositionally as the statement that all of Bo's desire worlds have this property.

(30) λw PRO to hire$_w$ [$_{DP}$ [$_{NP}$ [$_{DP}$ the number of cooks$_w$ Al hired$_w$]] [$_N$ cooks$_w$]]]

As long as predicates in intensional contexts can be evaluated in the actual world, then, the rich syntax approach captures the number transparent reading with no further stipulations. The modifier DP contributes the cardinality component, and number transparency arises if that component is evaluated in the actual world. We will see next that, in contrast, number transparency poses a challenge to the rich number approach.

3.2 Number transparency: rich numbers?

Under the rich number analysis, the opaque reading arises from the parse of the embedded clause in (31). With both *cooks* and *hired* evaluated in the desire worlds, the NDP in (31) contributes the domain component in (32), which references the cardinality component (33). The proposition expressed by (31) is true in w just in case in w Bo hires at least one member of the set (32). (31) is accordingly equivalent to (30) as interpreted in the rich syntax analysis. And so the opaque reading is again derived as intended, as the statement that the set of worlds denoted by (31) includes all of Bo's desire worlds.

(31) λw PRO to hire$_{w}$ [$_{DP}$ the [$_{NP}$ number of cooks$_{w}$ Al hired$_{w}$]]

(32) *Domain component:* $\{y|\ \text{cooks}(w)(y) \wedge |y| = (33)\}$

(33) *Cardinality component:* $\max\{n|\ \exists x \in \{z|\ \text{cooks}(w)(z) \wedge |z| = n\}[\text{hired}(w)(\text{Al})(x)]\}$

But the number transparent interpretation poses a challenge. Crucially, there is no available constituent that contributes the cardinality component by itself. To confirm that this does not leave a compositional path to the opaque reading, we observe that no choice of world arguments for the available predicates delivers the target meaning. With the lower *hire* evaluated in @ and the higher *hire* in w, just as in (29), only two options remain: the sole occurrence of *cooks* can be evaluated in w or in @, resulting in the structure in (34-a) and (34-b), respectively.

(34) a. λw PRO to hire$_{w}$ [$_{DP}$ the [$_{NP}$ number of cooks$_{w}$ Al hired$_{@}$]]
b. λw PRO to hire$_{w}$ [$_{DP}$ the [$_{NP}$ number of cooks$_{@}$ Al hired$_{@}$]]

The parse in (34-a) derives the meaning given in (35) to (37). This reading differs from the number transparent reading only in the cardinality component. The cardinality component of the number transparent reading is given in (38). When compared to it, the predicted cardinality reading in (37) is "under-transparent", as it applies *cooks* to the desire world w, and, therefore, the lower bound on the number of cooks that Bo wants to hire is not determined by a set of actual cooks, as in the target reading.

(35) $\text{Acc} \subseteq \{w|\ \exists x \in (36)\ [\text{hire}(w)(\text{Bo})(x)]\}$

(36) *Domain component:* $\{y|\ \text{cooks}(w)(y) \wedge |y| = (37)\}$

(37) *Cardinality component:* $\max\{n|\ \exists x \in \{z|\ \mathbf{cooks(w)}(z) \wedge |z| = n\}[\text{hired}(@)(\text{Al})(x)]\}$

(38) *Cardinality component of number transparent reading:*
$\max\{n|\ \exists x \in \{z|\ \mathbf{cooks(@)}(z) \wedge |z| = n\}[\text{hired}(@)(\text{Al})(x)]\}$

On the other hand, the parse in (34-b) derives the meaning given in (39) to (41). This time, the choice of world arguments yields the cardinality component of the number transparent reading, but a different domain component. As we can see by comparing the predicted domain component in (40) to the target domain component of the number transparent interpretation in (42), the predicted domain component applies *cooks* to the actual world instead of w. So the overall meaning is "over-transparent": it requires Bo's desires to be about actual cooks only.

(39) $\text{Acc} \subseteq \{w|\ \exists x \in (40)[\text{hire}(w)(\text{Bo})(x)]\}$

(40) *Domain component:* $\{y|\ \mathbf{cooks(@)}(y) \wedge |y| = (41)\}$

(41) *Cardinality component:* $\max\{n|\ \exists x \in \{z|\ \text{cooks}(@)(z) \wedge |z| = n\}[\text{hired}(@)(\text{Al})(x)]\}$

(42) *Domain component of the number transparent reading:* $\{y|\ \mathbf{cooks(w)}(y) \wedge |y| = (41)\}$

4 Conclusion

Under the rich syntax analysis, entity predication with a NDP invokes an interpreted structure with a cardinality-denoting DP that modifies the NP. Number transparency diagnoses the presence of this DP in the interpreted structure. The next step is to clarify the syntactic principles that can yield this structure, including the covert duplication of the entity noun.

Acknowledgements. We gratefully acknowledge financial support from the Social Sciences and Humanities Research Council of Canada (Insight Grants 435-2018- 0524, and 435-2023-0146, PI: Alonso-Ovalle; and 435-2019-0143, PI: Schwarz). Our names are in alphabetical order.

References

Alonso-Ovalle, Luis and Bernhard Schwarz (2024). "Structural Ambiguity in Pseudo-Partitives: the Case of Quantity Nouns". In: *Proceedings of the Thirthieth-Third Semantics and Linguistic Theory Conference*. Ed. by Juhyae Kim et al. Cornell University: CLC Publications, pp. 603–621. DOI: `https://doi.org/10.3765/dxn8d863`.

Bale, Alan (2006). "The Universal Scale and the Semantics of Comparison". PhD thesis. McGill University.

Carlson, Gregory (1977). "A Unified Analysis of the English Bare Plural". In: *Linguistics and Philosophy* 1, pp. 413–457. DOI: `https://doi.org/10.1007/BF00353456`.

Chierchia, Gennaro (1998). "Reference to Kinds across Languages". In: *Natural Language Semantics* 6.4, pp. 339–405. DOI: `https://doi.org/10.1023/A:1008324218506`.

Cresswell, Max J (1976). "The Semantics of Degree". In: *Montague Grammar*. Ed. by Barbara H Partee. New York: Academic Press, pp. 261–292.

Cresti, Diana (1995). "Extraction and Reconstruction". In: *Natural Language Semantics* 3, pp. 79–122. DOI: `https://doi.org/10.1007/bf01252885`.

Dummett, Michael (1991). *Frege. Philosophy of Mathematics*. NY: Duckworth.

Hackl, Martin (2000). "Comparative Quantifiers". PhD thesis. Massachusetts Institute of Technology.

Hatcher, William S. (1990). *The Logical Foundations of Mathematics*. NY: Pergamin Press.

Klein, Ewan (1980). "A Semantics for Positive and Comparative Adjectives." In: *Linguistics and Philosophy* 4, pp. 1–45. DOI: `https://doi.org/10.1007/BF00351812`.

Moltmann, Friederike (2013). "Reference to Numbers in Natural Language". In: *Philosophical Studies* 162, pp. 499–536. DOI: `https://doi.org/10.1007/s11098-011-9779-1`.

Percus, Orin (2000). "Constraints on Some Other Variables in Syntax". In: *Natural Language Semantics* 8, pp. 173–229. DOI: `https://doi.org/10.1023/A:1011298526791`.

Rullmann, Hotze (1995). "Maximality in the Semantics of Wh-Questions". PhD thesis. Amherst, MA: University of Massachusetts, Amherst.

Scontras, Gregory (2017). "A New Kind of Degree". In: *Linguistics and Philosophy* 40, pp. 165–205. DOI: `10.1007/s10988-016-9200-9`.