# Dutch Books, Indicative Conditionals, and Rational Updating

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**Abstract**

Intuitive credence judgments in cases involving indicative conditionals sometimes seem to conflict with the claim that you should update your credences by conditionalization. For this very reason, however, these credence judgments appear Dutch-bookable. After all, a classic result of Lewis (1999) seems to show that, if you update your credences in any way that conflicts with conditionalization, then a cunning bookie can always lead you into a Dutch book situtation. In this paper, we argue that—despite appearances—the intuitive credence judgments in the relevant cases may not, in fact, be Dutch-bookable. The reason is that the Lewisian Dutch book argument for conditionalization turns out to rest on a hidden assumption. We drop that assumption, and give more general Dutch book results. Our results vindicate conditionalization only in special cases.

## 1 Introduction

Suppose we just rolled a six-sided die.[1] The only relevant information you have is that it's fair, and that it landed on one of its sides. Consider these sentences:

(1) The die landed on an even number. (Even)

(2) If the die landed on a low number (i.e., 1, 2, or 3), then it landed on 2. (Low $\to$ 2)

What are your credences in (1) and (2), respectively?

Answer: 1/2 and 1/3. In the first case, you know the die is fair, and you know that half of its sides are even. In the second case, you know there are three low numbers, each of them equally likely to come up, and only one of them is 2.

Next, suppose you learn that the die did, in fact, land even. Then, having learned this (and nothing stronger), what's your credence *now* in the sentence (2)?

Answer: 1. After all, if the die landed even, then it landed on 2, 4, or 6. So, *if* it also landed on a low number, that singles out one possibility in particular: it landed on 2.

These judgments are very natural. However, there appears to be a problem with them: they seem to leave you open to a Dutch book. To see why, suppose you value dollars linearly (with the obvious choice of units), and your credences match your fair betting quotients. Then, given the credence judgments we outlined above, it seems like the following bets should be acceptable to you, at a time before learning that the die landed even:

1. A \$25 on Even, which you'll sell for a price of \$11;

2. A \$30 bet on Low $\to$ 2, which you'll either buy or sell for a price of \$10.

Similarly, if you become certain of (2) after learning Even, then this bet should be acceptable to you as well:

[1]The case that follows is due to Ciardelli and Ommundsen 2022. For similar cases, see, e.g., Goldstein and Santorio 2021, Santorio 2022, and Khoo 2022.

3. A \$30 bet on Low $\rightarrow$ 2 that you'll buy for a price of \$25.

But if this is right, then it seems like a cunning bookie can Dutch book you, using one of the strategies below.

First, suppose the bookie knows your fair betting quotients. Then, suppose she asks you for your fair price for a \$30 bet on the conjunction Even $\wedge$ (Low $\rightarrow$ 2). If this price is above \$10, then she'll be willing to sell you that bet, at your fair price, as well as buy from you a \$30 bet on Low $\rightarrow$ 2 for a price \$10. By assumption, you're willing to accept both. But if you do, you'll be led into a *synchronic Dutch book*:[2]

| | Even $\wedge$ (Low $\rightarrow$ 2) | Even $\wedge$ $\neg$(Low $\rightarrow$ 2) | $\neg$Even $\wedge$ (Low $\rightarrow$ 2) | $\neg$Even $\wedge$ $\neg$(Low $\rightarrow$ 2) |
|---|---|---|---|---|
| Bet 1 | $30 - x$ | $-x$ | $-x$ | $-x$ |
| Bet 2 | $-20$ | 10 | $-20$ | 10 |
| Net | $10 - x$ | $10 - x$ | $-20 - x$ | $10 - x$ |

Table 1: Your net payoff if you sell the bet on Even $\wedge$ (Low $\rightarrow$ 2) for \$$x$ with $x > 10$

On the other hand—and more important for our purposes—if your fair price for the bet on Even $\wedge$ (Low $\rightarrow$ 2) is no more than \$10, then the bookie can buy that bet from you for a price \$10, and sell you a \$25 bet on Even for \$11. Again, *ex hypothesi*, both offers are fair or favorable, by your lights. So, you should be willing to accept them both. But if you do, then the bookie will wait to find out if the die landed even. And if it does, she'll offer to sell you a \$30 bet on Low $\rightarrow$ 2 for \$25, which is again an acceptable bet to you, by assumption. However, these transactions lead you into a *diachronic Dutch book*:

| | Even $\wedge$ (Low $\rightarrow$ 2) | Even $\wedge$ $\neg$(Low $\rightarrow$ 2) | $\neg$Even |
|---|---|---|---|
| Bet 1 | 14 | 14 | -11 |
| Bet 2 | -20 | 10 | 10 |
| Bet 3 | 5 | -25 | 0 |
| Net | -1 | -1 | -1 |

Table 2: Your net payoff if you sell the bet on Even $\wedge$ (Low $\rightarrow$ 2) for \$10 (or less)

In this case, the reason seems to be that your credence judgments after learning diverge from the prescriptions of *conditionalization*.[3] And a famous result of Lewis (1999) seems to show that, if you update your credences in any way that conflicts with conditionalization, then this makes you Dutch-bookable.

Thus, we seem to have a puzzle. On the one hand, the credence judgments which our die case elicits seem intuitively rational.[4] On the other hand, they also appear to license betting dispositions that are Dutch-bookable and thereby rationally defective. So: what should we say in response? (Note also that the case discussed here is not unique. Analogous problems seem to arise in *many* cases involving indicative conditionals—i.e., sentences like our (2).)

In this paper, we're going to offer a partial diagnosis of this puzzle. In contrast to the Dutch book arguments we've just spelled out, our view is that the credence judgments elicited

[2]The reason for this, of course, is that in this case, your credences wouldn't satisfy Kolmogorov's probability axioms. In particular, having a fair price of more than \$10 for Even $\wedge$ (Low $\rightarrow$ 2) would imply that you're more confident in a conjunction than you are in either conjunct.

[3]Recall that conditionalization says the following: if $c$ is the probability function representing your credences at the current time, and $c_A$ is your credence function after learning $A$ (and nothing stronger), then, for any $B$, $c_A(B) = c(B \mid A) := \frac{c(A \wedge B)}{c(A)}$, provided that $c(A) > 0$. We spell out exactly how these judgents conflict with conditionalization in the last section, below.

[4]Indeed, several authors have defended the claim that they're intuitively rational—see, e.g., the references given in fn. 1.

by our die case may not be Dutch-bookable after all. To establish this, we'll show that the second Dutch book given above actually rests on a hidden assumption—namely, an assumption about how rational agents will evaluate *betting plans*. However, this assumption is controversial. And dropping it turns out to yield Dutch book results for a more general rule of rational updating—one that we call *conditional conditionalization*. This rule delivers the intuitive credence judgments in cases like our die case (at least given natural background assumptions). And it agrees with conditionalization when restricted to purely "factual" sentences (i.e., sentences that don't themselves contain conditionals).

## 2 Motivating the Proposal

Before we get to that, however, let's first consider a bad objection to the foregoing diachronic Dutch-book argument.[5] Thus, suppose someone complains: Table 2 is a bit misleading. That table shows that Bet 3 pays \$0 in cases where the die lands odd. But Bet 3 is an unconditional bet, and so has well-defined payoffs in those cases. In particular, you might win the bet in some cases where the die lands odd, and thereby collect a net *gain* of \$4. For this reason, then, we should reject the diachronic Dutch book argument against the intuitive credence judgments.

| | Even ∧ (Low → 2) | Even ∧ ¬(Low → 2) | ¬Even ∧ (Low → 2) | ¬Even ∧ ¬(Low → 2) |
|---|---|---|---|---|
| Bet 1 | 14 | 14 | -11 | -11 |
| Bet 2 | -20 | 10 | 10 | 10 |
| Bet 3 | 5 | -25 | 5 | -25 |
| Net | -1 | -1 | 4 | -26 |

Table 3: A Better Table?

This, clearly, is a bad objection, because the payoffs of Bet 3 in cases where the die lands odd are, in some sense, *irrelevant* to you—Bet 3 won't be offered to you in those cases in the first place. True, you can dream about the \$5 that you *could* win, *if* the bet *were* offered to you in those situations. But the fact is that you won't be offered that bet in those cases, and you know it. So what's the upshot? It's that you should take into account the payoffs of Bet 3 only *if* that bet is offered to you.

However, reflecting on this idea raises another concern about the foregoing Dutch book argument. To see what it is, first consider this sentence:

(3) If Bet 3 is offered, then you will win that bet.

Now, should you be certain of (3) *before* you learn whether the die landed even? To us, the answer isn't obvious. Indeed, there seem to be two plausible replies:

- *First Answer.* No, it's not rational for you to be certain of (3) before you learn whether the die landed even. To see why, notice that, given your background knowledge, Bet 3 is offered if and only if Even is true, and you win Bet 3 if and only if Low → 2 is true. By your own lights, it's possible for Even to be true and yet Low → 2 to be false, since you think the former is more probable than the latter. Thus, it's *not* rational for you to be certain of sentence (3).

[5] Due to considerations of space, we'll focus on the diachronic Dutch book argument here, i.e., the one in which your fair price for the bet on Even ∧ (Low → 2) is no more than \$10. Note also that, in a longer version of this paper, we discuss several other objections to this Dutch book argument. For example, we discuss an objection which says that the Dutch book is "practically defective", because Bet 2 cannot be "settled". We also discuss *contextualist* and *expressivist* responses to the puzzle. Again, see McNamara and Zhang MS for further details.

- *Second Answer*. Yes, it *is* rational for you to be certain of (3) before you learn whether the die landed even. Again, given your background knowledge, Bet 3 is offered if and only if Even is true; and you win Bet 3 if and only if Low → 2 is true. The sentence (3) is true just in case 'If the die landed even, then it landed on 2 if it landed on a low number' is true. (In symbols: Even → (Low → 2).) And intuitively, it seems rational for you to be certain of *that*.

We think both answers here are plausible. However, the Dutch book argument outlined in Table 2 *assumes* the first answer to our question. More specifically, it assumes that, before you learn whether Even is true, it's a live possibility for you that you might lose $25 if you take Bet 3.

However, suppose you endorse the second answer to our question, i.e., you think it's impossible for you to lose Bet 3 if it's offered and you take it. Then, it appears that you *won't* lose $1 for sure by accepting Bets 1-3. For if the die lands even, then you'll get $14 from Bet 1 and $5 from Bet 3, and you *might* win Bet 2 and keep a $10 premium from the bookie. As a result, there is a possibility in which you have a net profit of $29. Thus, taking into account the payoffs of Bet 3 in cases where the die lands odd is a bit like taking into account the payoffs we mentioned in the bad objection outlined above, if you think *Second Answer* is right:

| | Even ∧ (Low → 2) | Even ∧ ¬(Low → 2) | ¬Even ∧ (Low → 2) | ¬Even ∧ ¬(Low → 2) |
|---|---|---|---|---|
| Bet 1 | 14 | 14 | -11 | -11 |
| Bet 2 | -20 | 10 | 10 | 10 |
| Bet 3 | 5 | 5 | 5 (✗) | -25 (✗) |
| Net | -1 | 29 | -1 | -1 |

Table 4: The Best Table?

We can make this point more general. According to Dutch book results like the one given by Lewis (1999), the core of a diachronic Dutch book argument is the observation that a *plan* for how to accept or decline certain bets is payoff-equivalent to certain compounds of unconditional bets. As Lewis describes it:

> Consider a conditional bet: that is, a bet that will be null and void unless its condition is met. We note, first, that the conditional bet is equivalent in its outcome, come what may, to a certain pair of unconditional bets. We note, second, that the conditional bet is also equivalent in its outcome, come what may, to a certain contingency plan whereby one's future betting transactions are made to depend on the arrival of new evidence. The first equivalence yields a well-known synchronic argument relating the prices of conditional and unconditional bets. The second equivalence yields a diachronic argument relating the present prices of conditional bets to the future prices, after various increments of evidence, of unconditional bets. (p. 404)

Thus, in our example, the outcome of taking Bet 3 if the die landed even is equivalent to the outcome of buying a $30 bet on Even ∧ (Low → 2) for $17.5 and selling a $25 bet on Even for $12.5—or at least it is, according to Lewis. However, this compound of unconditional bets is unacceptable to you since, by assumption, $12.5 is the minimum price at which you are willing to sell a $25 bet on Even, and $17.5 is above the maximum price at which you are willing to buy a $30 bet on Even ∧ (Low → 2). This is the sense in which you "hold two contradictory opinions about the expected value of the very same transaction", according to Lewis: you find the contingency plan acceptable, but the relevant pair of unconditional bets unacceptable, even though, *ex hypothesi*, they are the same.

What we are suggesting, however, is that whether the two betting arrangements really *are* the same depends on what you think will happen *if* the die ultimately lands on even. In turn, this depends on your interpretation of the conditional 'If Bet 3 is offered, then you will win that bet'. As we argued above, there's a reasonable interpretation of 'if' such that it's rational for you to be certain *ex ante* that this conditional is true. And if you *are* certain of this conditional, then, by your own lights, the plan of buying Bet 3 for \$25 if the die lands even will *not* be payoff-equivalent to buying a \$30 bet on $\mathsf{Even} \wedge (\mathsf{Low} \rightarrow 2)$ for \$17.5 and selling a \$25 bet on Even for \$12.5; the former won't lose you money, but the latter might. Thus, from this perspective, the Dutch book argument above doesn't show that you hold contradictory opinions about the expected value of the very same transaction. Instead, from your perspective, the two transactions are very different.

## 3 Conditional Conditionalization

In this section, we're going to try making that insight more precise. To do so, let's start by considering yet another sentence:

(4) If the die landed on an even number, then it landed on 2 if it landed on a low number. $(\mathsf{Even} \rightarrow (\mathsf{Low} \rightarrow 2))$

Now, what's your credence in sentence (4)?

Intuitively, the answer here is '1', and the reasoning is similar to the reasoning we gave for certainty in (2) after learning Even: the only even number that's low is 2. So, if the die landed even; and if, in addition, it landed on a low number; then it must have landed on 2—there are no other (relevant) possibilities.

If this is right, however, then it suggests the rationality of being certain of $\mathsf{Low} \rightarrow 2$ after learning Even is closely related to the rationality of being certain of $\mathsf{Even} \rightarrow (\mathsf{Low} \rightarrow 2)$ prior to this learning event.[6] With this in mind, then, we want to propose that the connection is captured by the following rule of rational learning:

> **Conditional Conditionalization**. After learning the truth of $A$ (and nothing stronger), your new credence in any $B$ should be equal your old conditional credence in the indicative conditional $A \rightarrow B$, conditional on $A$. Formally, if $c$ is your prior credence function, and $c_A$ is your posterior credence function, after learning $A$ (and nothing stronger), then, if $c(A) > 0$:
>
> $$c_A(B) = c(A \rightarrow B \mid A) := \frac{c(A \wedge (A \rightarrow B))}{c(A)}.$$

Notice that, if $c(A \wedge (A \rightarrow B)) = c(A \wedge B)$, then this rule *entails* classical conditionalization (according to which $c_A(B) = c(B \mid A)$). On most theories of conditionals, moreover, this equality—known as *probabilistic modus ponens*—holds when both $A$ and $B$ are "factual" (i.e., they're sentences that do not themselves contain conditionals).[7] Thus, to this extent, classical conditionalization (restricted to factual sentences) can be viewed as a special case of conditional conditionalization.

On the other hand, the two rules come apart when extended to non-factual sentences. In particular, they disagree about the correct probability judgments in our die case. To see this,

[6] In turn, the rationality of being certain of $\mathsf{Even} \rightarrow (\mathsf{Low} \rightarrow 2)$ might depend on your preferred semantics for indicative conditionals. For example, on the "path semantics" for indicative conditionals given by Goldstein and Santorio 2021 and Santorio 2022, it's natural to think that certainty in this sentence is required.

[7] Note also that it holds for any semantic theory that validates plausible "centering" principles—even in the case in which $A$ and $B$ are non-factual.

suppose $c$ is a classical probability function such that $c(\mathsf{Even} \wedge (\mathsf{Low} \to 2)) \leq c(\mathsf{Low} \to 2)$. Then, according to conditionalization, if $c(\mathsf{Even}) = 1/2$ and $c(\mathsf{Low} \to 2) = 1/3$, then:

$$c_{\mathsf{Even}}(\mathsf{Low} \to 2) = \frac{c(\mathsf{Even} \wedge (\mathsf{Low} \to 2))}{c(\mathsf{Even})} \leq \frac{1/3}{1/2} < 1.$$

On the other hand, conditional conditionalization vindicates the intuitive credence judgments the die case, provided that you are *ex ante* certain of the conditional $\mathsf{Even} \to (\mathsf{Low} \to 2)$. For, if you are certain of that conditional, then according to conditional conditionalization, your posterior credence in $\mathsf{Low} \to 2$ after learning $\mathsf{Even}$ should be:

$$\frac{c(\mathsf{Even} \wedge (\mathsf{Even} \to (\mathsf{Low} \to 2)))}{c(\mathsf{Even})}.$$

Then, since you are certain of $\mathsf{Even} \to (\mathsf{Low} \to 2)$, the numerator equals $c(\mathsf{Even})$, which means that the ratio equals 1. Crucially, moreover, this entailment holds for *any* positive credal assignments to $\mathsf{Even}$ and $\mathsf{Low} \to 2$.

Thus, to show that the intuitive credence judgments are not Dutch-bookable, it suffices for us to show that conditional conditionalization is not Dutch-bookable. As we hinted, however, in order to do this, we need to reject Lewis's assumption that the plan of taking a bet on $B$ if $A$ is true is payoff-equivalent to certain compounds of unconditional bets on $A \wedge B$ and $A$, respectively. The good news, however, is that, in the present case, this assumption seems questionable. After all, if you think you *won't* lose the bet on $\mathsf{Low} \to 2$ if you take it when $\mathsf{Even}$ is true—since you're *ex ante* certain of $\mathsf{Even} \to (\mathsf{Low} \to 2)$—then, by your lights, this plan is not payoff-equivalent to any compounds of unconditional bets on $\mathsf{Even} \wedge (\mathsf{Low} \to 2)$ and $\mathsf{Even}$.

How, then, should you evaluate the payoffs of betting on $\mathsf{Low} \to 2$ if $\mathsf{Even}$ is true? Here is a natural thought: you should evaluate the payoffs of this plan in the same way you evaluate a conditional bet on $\mathsf{Even} \to (\mathsf{Low} \to 2)$, *conditional on the truth of Even*. More generally:

> **Plans-as-Conditional-Bets-on-Conditionals (PCBC)**. A plan of *taking a bet on $B$ if $A$ is true* is payoff-equivalent to a conditional bet on the conditional $A \to B$, conditional on $A$.

PCBC entails Lewis's assumption that betting plans are payoff-equivalent to conditional bets, under the assumption that a conditional bet on $B$ given $A$ is payoff-equivalent to a conditional bet on $A \to B$ given $A$. Again, this assumption is plausible if both $A$ and $B$ are factual, for then both bets are payoff-equivalent to a conditional bet on $A \wedge B$, given $A$. On the other hand, this assumption plausibly fails if $B$ contains conditionals. And as we saw, these are also precisely cases where Lewis's assumption seems problematic. Thus, we have the following result:[8]

**Theorem 3.1.** *Suppose an agent is not synchronically Dutch-bookable and satisfies PCBC. Then she is not diachronically Dutch-bookable if and only if she updates by conditional conditionalization.*

Once again, this result entails classical conditionalization, given probabilistic *modus ponens*. Relatedly, our Dutch book theorem entails the Dutch book theorem of Lewis (1999) for classical conditionalization, if we assume a conditional bet on $A \to B$ given $A$ is payoff-equialent to a conditional bet on $A \wedge B$ given $A$.[9] As we heard, however, it's not uncontroversial that this should be the case. And if we reject Lewis's assumption, then the intuitive credence judgments with which we started may not be Dutch-bookable after all.

[8] A proof of this result can be found in McNamara and Zhang MS.

[9] Incidentally, our main result also entails the Dutch book result of McGee (1989), given a different, and competing, assumption about the probabilities of conditionals—namely, that $A$ and $A \to B$ are probabilistically independent. See McGee's paper for further discussion.

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