Prover-Skeptic Games and Logical Pluralism*

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Abstract

Logical Pluralists face an explanatory challenge: how is it that there are a multitude of different, equally correct, accounts of logical consequence. In this paper we will argue that this challenge can be met quite easily if one adopts the 'built-in opponent' conception of logic: a multi-agent, dialogical view of the nature of logical consequence. We introduce Prover-Skeptic games in order to model this view of logical consequence, and use our formal models to make clear how a certain kind of pluralism about explanation leads to an interesting variety of logical pluralism.

1 The Explanatory Problem for Logical Pluralism

Is there a single answer to the question of whether a given argument is (deductively) valid? According to *logical monists* the answer here is an emphatic 'yes', deductive validity being determined by the deliverances of their favoured logic. According to *logical pluralists* such as JC Beall & Greg Restall ([2]) and Stewart Shapiro ([13]) the answer to this question is 'no'; there are a multitude of different logics which determine deductive validity. Immediately the logical pluralist faces a problem, raised quite elegantly by Rosanna Keefe in the following quote:

> A characterisation of a pluralist position needs to explain what it is to endorse all of the various consequence relations the pluralist accepts and how they relate to an intuitive notion of logical consequence. [10, p.1376]

Call this the *explanatory challenge* for logical pluralism. The logical pluralist needs (*i*) to explain in what sense these logics are all correct accounts of deductive validity, i.e. what it means to 'endorse' these logics; and (*ii*) explain how these different logics relate to an intuitive notion of logical consequence. The main concern in [10] (esp. in section 3 therein) is to show that the answer to this challenge given by Beall & Restall in [2] is inadequate, and we shall not detain ourselves any further with it here. Instead we will take a different tack. What we will argue for here is the claim that a particular dialogical conception of logic and deduction, the built-in opponent conception of deduction (outlined in §2) allows quite naturally for a limited but interesting form of logical pluralism. In order to show this we will make use of a formalisation of this dialogical conception of deduction in terms of a novel kind of dialogue game (§3), using this to argue in §5 for a particular kind of logical pluralism which is able to easily answer the explanatory challenge.

2 The Built-In Opponent Conception of Deduction

Here we will adopt the particular multi-agent dialogical conception of logic defended by Catarina Dutilh Novaes in [4, 5]—the *built-in opponent conception of deduction.* According to this approach rules of inference reflect the rules for engaging in a certain kind of discursive practice.

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The discursive practices involved here are a specialised variety of semi-adversarial dialogues, the participants of which have opposite goals: the *Prover* (or Proponent) seeks to establish that a certain conclusion follows from given premises, while the *Skeptic* (or Opponent) seeks to block the establishment of this conclusion. The adversarial nature of this dialogues is obvious, and flows from the players having opposite goals. At a higher-level, though, there is also a large degree of cooperation between the two players, as Catarina Dutilh Novaes explains:

> Proponent's job is not only to 'beat Opponent'; she also seeks to *persuade* Opponent of the truth of the conclusion, if he has granted the truth of the premises. In fact, the goal is not only to show *that* the conclusion follows from the premises, but also *why* it does; this corresponds to the idea that deductive arguments ought to have explanatory value. In this sense, Proponent and Opponent are cooperating in a common inquiry to establish what follows from the premises, and thus to further investigate the topic in question. [5]

It is not only Prover's role which has a cooperative component, the Skeptic's job also requires a level of higher-order cooperation, requiring them to not simply obstinately refuse that conclusion follows from the premises, but instead to say what would be required to convince them that the conclusion followed from the premises. We will say more about this in §3.

Contemporary logical practice does not, at first glance, resemble the picture we have been sketching above of logic involving a kind of multi-agent interaction. Typically we think of the act of determining whether an argument is deductively valid as being a rather solitary activity, and this thought is at least partially correct. The other important aspect of the present approach is the idea that over time Skeptic has been progressively 'silenced' and idealised, until they are no longer an active participant but instead are part of the deductive method itself—their role being internalised and, in essence, played 'offline' by Prover. To beat such an idealised opponent Prover needs to make sure that there are no counterexamples to the inferential steps which they provide as responses to the Skeptic's challenges. Thus the internalisation of Skeptic provides a bridge between multi-agent dialogical practices and mono-agent inferential practices.

3 Prover-Skeptic Games

In order to formally model the built-in opponent conception of deduction we will use *Prover-Skeptic Games*, a kind of dialogue game introduced in [16, p.89–94] (where they are used to game-theoretically characterise the implicational fragment of intuitionistic logic), although we will largely follow the presentation of such games given in [14] (where they are used to characterise the associative Lambek calculus). In order to get a feel for how these games work consider the following dialogue.

A Dialogue

In the library two logic students, ***Penelope*** *and* ***Scott*** *are arguing over the validity of the argument from $p \to q$ and $q \to r$ to $p \to r$. Penelope thinks that this argument is valid and is trying to convince Scott, who is skeptical.*

- **Penelope (1)**: I reckon that $p \to r$ follows from $p \to q$ and $q \to r$
- **Scott (1)**: Yeah? Well if that's so then suppose I grant you $p \to q$ and $q \to r$ along with p, how are you meant to get r?
- **Penelope (2)**: If you grant me q I can get r from $q \to r$ (which you just granted).
- **Scott (2)**: But why should I grant you q?

Axioms

$$A \succ A$$

Introduction/Elimination Rules

$$\frac{\Gamma, A \succ B}{\Gamma \succ A \to B}\ (\to I) \qquad \frac{\Gamma \succ A \quad \Delta \succ A \to B}{\Gamma, \Delta \succ B}\ (\to E)$$

Structural Rules

$$\frac{\Gamma, A, A \succ B}{\Gamma, A \succ B}\ (W) \qquad \frac{\Gamma \succ B}{\Gamma, A \succ B}\ (K)$$

Figure 1: A Natural Deduction System, in Sequent-to-Sequent style, for the implicational fragment of Intuitionistic Logic, where the structural rules *contraction* ((W)) and *weakening* ((K)) are explicit. (Note that in ($\to E$) that $A \to B$ is the major premise, and A the minor premise.)

- **Penelope (3)**: Well if you were to grant me p then I could get q from $p \to q$ which you granted at the start.
- **Scott (3)**: But why should I grant you p?
- **Penelope (4)**: Because you granted it to me at the start!

Penelope leaves the library triumphantly.

The above dialogue has the structure of the kind of dialogical interactions which are at the heart of the built-in opponent conception of deduction, with Penelope (as her name mnemonically suggests) acting as Prover and Scott as Skeptic. This is also the structure of Prover-Skeptic games. In [16] these are defined in austere syntactic terms, but we will find it much more helpful to characterise these games directly in terms of a natural deduction proof system like that given in Figure 1.[1]

Before we go on to describe the games themselves, let us dispense with some notational preliminaries. Throughout we will be concerned with the language $\mathcal{L}$ of implicational logic, the formulas of which are constructed out of a countable supply of propositional variables $p_0, p_1, p_2, \ldots$ (the first three of which we abbreviate as p, q, r) using the binary connective '$\to$' of implication. Throughout we will use uppercase roman letters as schematic letters for formulas from $\mathcal{L}$, and uppercase Greek letters for multisets of formulas from $\mathcal{L}$. Recall that a *multiset* is a set in which elements can occur multiple times, making $[A, A, B]$ and $[A, B]$ distinct multisets (we use $[A_1, \ldots, A_n]$ to denote the multiset consisting of the formulas $A_1, \ldots A_n$).[2] A *sequent* is a pair $\langle \Gamma, A \rangle$ of a multiset of formulas Γ and a formula A, which we will write throughout as $\Gamma \succ A$.

[1]Natural deduction systems are rarely given in this 'structurally explicit' sequent-to-sequent style, the most notable exception being the treatment of intuitionistic logic in [3, p.88f] which has an explicit rule of weakening, or the systen *Nat* in [9, p.114], in which it is shown that (K) (there called $\mathbb{M}$) is derivable given the rules for conjunction. In both of these cases the need for an explicit rule of contraction is avoided by working with sets rather than multisets. Usually structural variation is captured in natural deduction systems in terms of different policies regarding how assumptions can be discharged, but we will find it particularly helpful to be fully explicit about the applications of structural rules throughout.

[2]For more information on multisets consult [15]

3.1 Defining the Game

Given a multiset of sequents $\mathfrak{S}$ and sequents β, γ let us say that $\beta \Rightarrow^*_{(I)} \gamma$ iff we can derive γ from β via zero or more applications of our introduction rules (in our case, $(\rightarrow I)$), and $\mathfrak{S} \Rightarrow^*_{(E)} \beta$ iff we can derive β from the sequents in $\mathfrak{S}$ via zero or more application of our elimination rules and structural rules (in our case, $(\rightarrow E)$, (W) and (K)). Finally say that a sequent $\Gamma \succ A$ is *atom-focused* just when A is a propositional atom.

Definition 3.1 (Dialogue Game)**.** *A* dialogue *over* (Γ, A) *is a (possibly infinite) sequence* $\mathfrak{S}_1, \beta_1, \mathfrak{S}_2, \beta_2, \ldots$ *where each* $\mathfrak{S}_i$ *is a multiset of sequents, and each* β_i *is a sequent where:*

1. $\mathfrak{S}_1 = [\Gamma \succ A]$
2. $\beta_i \Rightarrow^*_{(I)} \sigma$ *for some non-axiomatic* $\sigma \in \mathfrak{S}_i$ *and some atom-focused sequent* β_i
3. $\mathfrak{S}_{i+1} \Rightarrow^*_{(E)} \beta_i$

In this setting it is helpful to think of Prover and Skeptic moves in the following terms. First, read a sequent $\Gamma \succ A$ in our dialogues as claiming that 'you cannot grant me Γ without also granting me A', then:

- A *Skeptic move* β where $\beta \Rightarrow^*_{(I)} \sigma$ can be interpreted as questioning σ by asking Prover to show β. That is to say, such a move involves questioning Prover's claim that if you grant me Γ you must also grant me A (where $\sigma = \Gamma \succ A$) by asking Prover to demonstrate that if they grant Γ' that they must also grant p_β, a sequent from which we can derive σ.
- A *Prover move* $\mathfrak{S}$ in responce to a Skeptic challenge β can be interpreted as providing grounds for the claim that if they are granted Γ' then they must also be granted p_β, as if they are granted $\Gamma_1, \ldots, \Gamma_n$ then they must also be granted $A_1, \ldots, A_n$ (where $\mathfrak{S} = [\Gamma_1 \succ A_1, \ldots, \Gamma_n \succ A_n]$). Note that given the structure of our elimination rules the multiset union of the Γ_is will be a sub-multiset of Γ', and so the collection of sequents in $\mathfrak{S}$ can be seen as recording how it is that if they are granted Γ' they must also be granted p_β.

Moves in Prover-Skeptic games correspond quite naturally to the kinds of actions taken by Prover/Proponent and Skeptic/Opponent in the kind of semi-adversarial dialogues which are at the heart of the built-in opponent conception of deduction. What is just as important, though, is that these dialogue games also make clear how it is that Skeptic can be internalised to the method itself. In particular, we can give a completely deterministic, and syntactic, characterisation of the structure of a Skeptic challenge.

Lemma 3.2. *Suppose that* $\sigma = \Gamma \succ A$*, and:*

- $A = B_1 \rightarrow (\ldots (B_{n-1} \rightarrow (B_n \rightarrow p_i)) \ldots)$
- $\Gamma' = [B_1, \ldots, B_n]$.

Then $\beta_i \Rightarrow^*_{(I)} \sigma$ *iff* $\beta_i = \Gamma', \Gamma \succ p_i$.

This mean that Skeptic moves do not require any creativity to perform, and are thus the kind of moves which are apt to be internalised and 'simulated offline' by Prover. This fits quite well with the central idea of the built-in opponent conception that the Skeptic is not an active participant in these dialogues, as their role can simply be played by Prover (with the aid of a randomising device to choose the sequent to be challenged).

How do these games connect up to logical consequence? The answer here is the standard one in dialogical logic: A is a logical consequence of Γ when Prover has a winning strategy in the appropriate dialogue game. That is to say, that no matter what moves Skeptic makes, Prover can always win any dialogue over (Γ, A), where Prover wins a dialogue if it reaches a point at which Skeptic can make no further move. Similarly, we can say that Skeptic wins if Prover can make no further move.[3] This does not necessarily cover all the cases, though, as we could end up in a situation where both players always have moves available to them. What is important for present purposes, though, are the dialogues for which Prover has a winning strategy.

Definition 3.3 (Winning Strategy). *A* winning strategy *(for Prover) for the dialogue over* (Γ, A) *is a labelled tree where*

- *The root note of the tree is a P-node* $([\Gamma \succ A])$*.*
- *Each branch is a dialogue over* (Γ, A)
- *Every P-node* $(\mathfrak{S}_i)$ *has* $|\mathfrak{S}_i|$ *S-node descendants.*
- *Every S-node has a single P-node descendant.*

In [16] it is shown that Prover has a winning strategy in the dialogue over (Γ, A) iff $\Gamma \succ A$ is valid in the implicational fragment of intuitionistic logic. We will discuss similar adequacy results below in §5.

3.2 An Example

To make clear how this works, let us now formalize the dialogue we gave at the opening of this section. This is a dialogue over $([p \to q, q \to r], p \to r)$, and proceeds as follows.

P(1)	$[p \to q, q \to r \succ p \to r]$	*Prover Starts*
S(1)	$p \to q, q \to r, p \succ r$	*Skeptic challenges the sole Prover assertion.*
P(2)	$[q \to r \succ q \to r, \quad p \to q, p \succ q]$	*Prover replies, offering sequents from which the Skeptic's challenge can be derived using* $(\to E)$*.*
S(2)	$p \to q, p \succ q$	*Skeptic challenges the only non-axiomatic sequent.*
P(3)	$[p \to q \succ p \to q, p \succ p]$	*Prover replies, winning the dialogue as Skeptic has no further moves available to make.*

Moreover, given that at no stage could Skeptic have made any other move, the above dialogue corresponds quite directly to a winning strategy for Prover.

4 Proofs and Explanations

The particular discursive practices which are at the heart of the built-in opponent conception of deduction are clearly norm governed. Some of these norms are quite simple and obvious: for example, the players are normatively required to take turns, and Skeptic is obliged to only doubt or query claims which are not obviously correct. In fact, the two norms mentioned are partially constitutive of the practice involved (and are mirrored in Prover-Skeptic games by the turn taking requirement, and the ban on Skeptic challenging axiomatic sequents). There

[3] As presented here Skeptic is rather more 'tolerant' than is ideal—allowing Prover to offer any collection of sequents they want, so long as they derive the challenge sequent. In part this means that there are no finite winning conditions for Skeptic, as Prover will always have a response open to them. This is not that important for present purposes, but does represent one degree in which the current presentation does not quite match up with the built-in opponent conception of deduction.

is another normative aspect of this practice which is worth singling out, though. Namely that both sides must agree on what counts as an adequate response to a challenge.

Recall that Prover moves are meant to do more than merely show that the premises follow from the conclusion, they're meant to be *explanatory*—answers to the question 'why does this follow'. As is pointed out in [7] what counts as an answer to a why question varies with various features, most importantly here with the interests of the person asking the question. This means that what counts as an adequate response to a challenge will be relative to the interests of the audience for which the proof is intended to be explanatory—i.e. to the interests of a community of inquiry.

We can see an example of this in work on the foundations of geometry at the beginning of the 20th century. There mathematicians appear to be concerned not just with what axioms are used in proving a theorems, but also in the *number of times* such axioms are, or indeed must, be appealed to in proving it. For example, G. Hessenberg in 1905 proves the Desargues axiom follows from a collection of axioms for plane projective geometry along with a threefold use of the Pappus axiom. In [11] it is also noted that all known proofs both require, and explicitly mention, this threefold use. One way (and I will readily admit, not the only way) of understanding this situation is in terms of a change in the norms governing deductive dialogues. Rather than it simply being sufficient that you show that the challenged claim follows 'of necessity' from the claims provided, the Prover must also indicate the extent to which different claims are used in showing this—resulting in a change in the norms of explanation in this context. One way to register this 'resource consciousness' is by working in a system in which we can draw a distinction between when A follows from Γ and two copies of B, and when A follows from Γ and a single copy of B—i.e. work in a system in which we do not have the rule (W) of contraction.

This gives us an example, in mathematics of all places, where the norms which govern deductive dialogues appear to be (contra [2, p.88]) relative to communities of inquiry, or perhaps better, relative to the particular interests of those communities. So, in this case, a concern over the extent to which a given axiom must be used in a proof (perhaps sparked by foundational concerns about the truth of such axioms) leads to a particular norm governing the practice of proving—namely that you should note the number of times various axioms must be used in proofs.

Similarly, one might argue that requests for explanatory proofs point towards a kind of relevance criterion, perhaps one in the style of the relevant logic á la [1] where one drops the structural rule of weakening $((K))$. This might suggest a more general norm which one might appeal to in deductive dialogues—of only providing the evidence that is *required* for deriving the challenge, rather than merely providing *enough* information to derive the challenge.

The resulting kind of pluralism has a very pleasant shape. On the dialogical conception described above, logic is normative for a certain kinds of semi-adversarial dialogue games (deductive dialogues). Given that such dialogues can be governed by different norms which can effect what counts as an acceptable responce in such games, we then get a kind of logical pluralism (a 'local pluralism' in the terminology of [8])—which logic is correct depends on the norms which govern that kind of deductive dialogue. The kind of pluralism which emerges here is similar to that argued for by Hartry Field in [6]. Field argues that the normativity of logic for thought combined with pluralism about epistemic norms gives rise to a form of logical pluralism, while here we have argued that the normativity of logic for certain kinds of dialogical interactions coupled with pluralism about the norms governing those dialogical interactions gives rise to logical pluralism. In both cases pluralism about logic arises out of a pluralism concerning what it is over which logic holds normative sway, good reasoning in Field's case and certain kinds of dialogues in ours.

5 A Route to Pluralism

This puts us in a comfortable position to deal with the explanatory challenge for logical pluralism. We have an intuitive notion of logical consequence characterised by the built-in opponent conception of deduction formalised using Prover-Skeptic games. In the previous section we argued that, in this framework, what counts as a response to a challenge is interest relative, and that this interest relativity can sometimes result in the inadmissibility of certain structural rules in Prover moves. What remains to be done is to show that this in turn gives rise to a plurality of *logics*. To do this we will find it helpful to define S-dialogues, generalising our previous definition of a dialogue. If $S \subseteq \{(W),(K)\}$ let us write $\mathfrak{S} \Rightarrow^*_{(E)S} \beta$ iff we can derive β from the sequents in $\mathfrak{S}$ via zero or more applications of our elimination rules and structural rules from S.

Definition 5.1 (S-Dialogue)**.** *Let* $S \subseteq \{(W),(K)\}$*. Then an* S-dialogue *over* (Γ, A) *is a (possibly infinite) sequence* $\mathfrak{S}_1, \beta_1, \mathfrak{S}_2, \beta_2, \ldots$ *where each* $\mathfrak{S}_i$ *is a set of sequents, and each* β_i *is a sequent where:*

1. $\mathfrak{S}_1 = \{\Gamma \succ A\}$
2. $\beta_i \Rightarrow^*_{(I)} \sigma$ *for some non-axiomatic* $\sigma \in \mathfrak{S}_i$
3. $\mathfrak{S}_{i+1} \Rightarrow^*_{(E)S} \beta_i$

If we also write $\vdash_S \Gamma \succ A$ to mean that there is a proof of $\Gamma \succ A$ which uses only structural rules in S then we have the following adequacy result.

Theorem 5.2 (Adequacy)**.** *Prover has a winning strategy in an* S*-dialogue over* (Γ, A) *iff* $\vdash_S \Gamma \succ A$.

The 'only if' direction of the above Theorem is relatively straightforward, being proved by relatively straightforward induction on the number of P-nodes in the strategy. This direction is vastly simplified by the fact that we have defined our dialogue games directly in terms of a natural deduction proof system. The 'if' direction is significantly more revealing. For reasons on space we will simply sketch how it goes here. Firstly note that any provable sequent $\Gamma \succ A$ has a proof which is in normal form in the sense of [12]. One of the distinctive features of normal form proofs is that they can be split up into 'tracks' ([17, p.143]), sequences of sequents which can be split into three parts: an E-part in which each sequent follows from the previous one via an E-rule or structural rules; a minimal sequent; and finally an I-part in which each sequent follows from the previous one using I-rules or structural rules. Given any proof in normal form we can then transform it into one in which: (1) the minimal sequent in each track is of the form $\Gamma \succ p_i$ for some Γ and p_i (putting it into $\beta\eta^-$-long form), and (2) structural rules are not used in the I-part of the track (by permuting the structural rules over the introduction rules). Call such a proof one in *top-heavy long normal form*. As it turns out, it is the fact that we can transform every proof into one in top-heavy long normal form which allows us to 'delay' all use of structural rules until Prover moves. Treating structural differences via different discharge policies (i.e. restricting multiple, or vacuous discharge) complicates matters here, as this would most naturally cause the presence or absence of structural rules to effect Skeptic moves, destroying the ability to internalize Skeptic in the natural way we are able to in the present setting.

Given an S-proof in top-heavy long normal form of the sequent $\Gamma \succ A$ it is a simple matter to read off a winning strategy for Prover in the S-dialogue over (Γ, A). Rather than explain the procedure in detail here we will present an illustrative example.

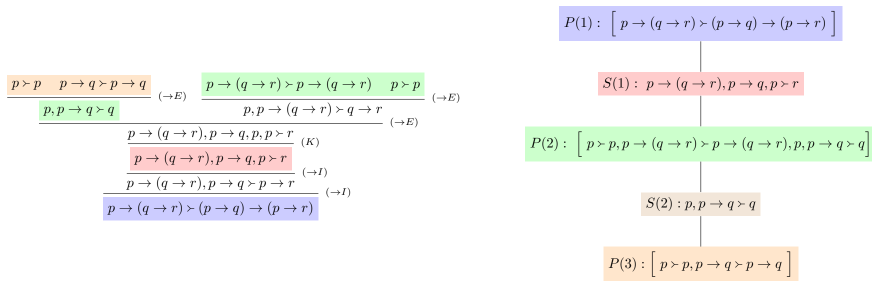

Figure 2: K-proof of $p \to (q \to r) \succ (p \to q) \to (p \to r)$ and a winning strategy in the K-dialogue over $(p \to (q \to r), (p \to q) \to (p \to r))$.

In Figure 2 we have a proof in top-heavy long normal form of the sequent $p \to (q \to r) \succ (p \to q) \to (p \to r)$. We can read a Prover winning strategy off this proof as follows. In essence Skeptic's challenge is the minimal formula on each track, and Prover responds by offering the sequent at the end of the track which the challenge is on along with the minor premises to any elimination rules used in the E-part of that track.

6 Conclusion

Logical pluralists face an explanatory challenge: they must explain what it is for a logic to be correct, and why there are many things which fit that bill. Here we have shown that a species of logical pluralism is forthcoming if we adopt the built-in opponent conception of deduction. On this approach a logic is correct if it is normative for the kind of dialogical interaction which one is currently engaged in, with different normative contraints on these interactions arising out of the interests of the participants. In many ways, though, the presentation is more suggestive than it is decisive. In particular:

- We have only dealt with a relatively limited logical vocabulary—pure implicational logic. In order to fully substantiate the claims made above we would need analogues of the above results for a more substantial selection of logical constants—at least $\{\wedge, \vee, \to, \neg\}$ in the propositional case.
- One might object that the above case we have made for the logical ramifications of the interest relativity of answers to questions about what follows from what could be stronger.

Future work is needed to bring this to fruition, but the above results do show that looking at logic from a dialogical perspective using Prover-Skeptic games affords a new and interesting view of traditional (and new) problems in philosophical logic.

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