An Interactive Approach to Proof-Theoretic Semantics

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Abstract

In truth-functional semantics for propositional logics, categoricity and compositionality are unproblematic. This is not the case for proof-theoretic semantics, where failures of both occur for the semantics determined by monological entailment structures for classical and intuitionistic logic. This is problematic for inferentialists, where the meaning of logical constants is supposed to be determined by their rules. Recent attempts to overcome these issues have primarily considered symmetric entailment structures, but these are tricky to interpret. Here, I instead consider an entailment structure that combines provability with the dual notion of disproof (or refutation). This is interpreted as a dialogue structure between the roles of prover and denier, where an assertion of a statement involves a commitment to its defence, and a denial of the statement involves a commitment to its challenge. The interaction between the two is constitutive of a proof-theoretic semantics capable of dealing with the above issues.

1 Inferentialism and categoricity

Logical inferentialism is usually taken to rest upon the idea that the meaning of a logical expression is determined by its inferential rules, where those rules have a substantive connection with ordinary inferential practices. Sometimes called proof-theoretic semantics, this approach should not take truth as the primary semantic notion, nor should it require access to truth-conditional semantics in order to fix the meanings of logical constants. So, the (fairly standard) distinction that is often made between formal proofs and semantic models, and upheld by soundness and completeness proofs, must be somehow overcome. This distinction rests upon the idea that formal proofs are purely syntactical entities, and, as such, can not be the sort of things that are meaningful in a true sense. In this respect, it is of central importance that inferentialism shows, instead, that patterns of inference are the sort of things that can be both represented syntactically, and confer meaning upon logical expressions. This coheres with the more general position in philosophy of language that draws upon Wilfrid Sellars' "inferential game of making claims and giving and asking for reasons", developed most prominently by Robert Brandom [3].[1]

One issue that arises for this approach to semantics is the well-known "categoricity" problem, where the standard inferential rules for classical logic fail to rule out non-standard interpretations, so they do not suffice to determine the meaning of logical constants. In particular, that framework is easily shown to be sound and complete with respect to *both* the classical semantic model, and a model in which every formula is interpreted "true". This is also problematic from the point of view of compositionality, since it is simple to show that the rules defining disjunction, for example, do not ensure that the truth-values of the sub-formulas for a formula $\alpha \vee \beta$ always determine the truth-value of that formula.

Say that a "logic", L, is an entailment structure, which is an ordered pair, $(S, \vdash_L)$, where S is a denumerable set of propositional formulas, and $\vdash_L$ is a binary entailment relation defined

[1]See also [11] for a defense of games as central to logic.

on $P(S) \times P(S)$ ($P(S)$ is the set of all finite subsets of S) satisfying the properties of reflexivity, transitivity, monotonicity, and finitariness. Entailment structures can be restricted in different ways, which we think of in terms of sequents in a logic. These are just ordered pairs, $\langle \Gamma, \Delta \rangle$ where Γ, Δ are finite (possibly empty) sequences of formulas of S. Then, say that a right-asymmetric sequent is restricted to at most a single formula on the right; a left-asymmetric sequent is restricted to at most a single formula on the left, and a symmetric sequent has no such restrictions. Now, say that a sequent rule $\mathcal{R}$ in any logic L is an ordered pair consisting of a finite sequence of sequent premises and a sequent conclusion $\mathcal{R} = \langle \{SEQ^P\}, SEQ^C \rangle$. In this way, we may think of a specific logic to be determined by a proof structure (set of axioms and sequent rules), where any collection of sequents $\mathcal{S}$ that is closed under the standard structural rules determines a finitary, normal, logic. For example, it is the case that $\Gamma \vdash \alpha$ iff for some finite $\Gamma_0 \subseteq \Gamma$, we have $(\Gamma_0 \vdash \alpha) \in \mathcal{S}$.[2]

Take the rules ordinarily used to define classical negation (in a right-asymmetric entailment structure):

$$\frac{\Gamma, \neg\alpha \vdash \beta \qquad \Gamma, \neg\alpha \vdash \neg\beta}{\Gamma \vdash \alpha}\ (Reductio) \qquad\qquad \frac{\Gamma \vdash \alpha \qquad \Gamma \vdash \neg\alpha}{\Gamma \vdash \beta}\ (EFQ)$$

Now, consider a standard Lindebaum-Asser construction: For any finite normal logic L, given a formula β, and a theory Γ (where $\Gamma, \beta \in S$) such that $\Gamma \nvdash \beta$, Γ can be extended to Γ' (where $\Gamma \subseteq \Gamma'$ and Γ' is relatively maximal with respect to β (in L) so that for no proper superset Γ'' of Γ' do we have $\Gamma'' \nvdash \beta$. In [2], it is proven that a semantics for a logic, L, is given by taking the characteristic function of relatively maximal theories for L as follows: given a sequent $\alpha_1, ..., \alpha_n \vdash \beta$ in a logic L, and a relatively maximal theory Γ' of L, we say that Γ' satisfies this sequent iff, whenever $\Gamma' \vdash \alpha$, for each $\alpha_1, ..., \alpha_n$, $\Gamma' \vdash \beta$. The issue mentioned above is that there are possible interpretations where both α and $\neg\alpha$ are in Γ', and which are not ruled out by the rules used above to determine the maximal sets of formulas. Similarly, the rules defining disjunction (in right-asymmetric form) ensure that whenever $\alpha \in \Gamma'$ or $\beta \in \Gamma'$, it is the case that $\alpha \vee \beta \in \Gamma'$. But, they fail to rule out a situation in which $\alpha \vee \beta \in \Gamma'$, whilst neither α nor β are.

An obvious fix for this issue, which also sheds light on the root of the problem, is to use a symmetric sequent structure to define the logic. The extension theorem in this context is slightly different since we can say that, for any finite normal logic L, and $\Gamma \not\mid\!\!\!\vdash\!\!\!\dashv \Delta$, Γ, Δ can be extended to $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, with $\Gamma' \not\mid\!\!\!\vdash\!\!\!\dashv \Delta'$ (and no proper superset $\Gamma'' \supseteq \Gamma'$, $\Delta'' \supseteq \Delta'$ do we have $\Gamma'' \not\mid\!\!\!\vdash\!\!\!\dashv \Delta''$). Let us call such maximal Γ', Δ' a quasi-partition. The semantics defined this way also differs: for a symmetric sequent, $\alpha_1, ..., \alpha_n \vdash\!\dashv \beta_1, ..., \beta_m$, in a logic L, and a quasi-partition Γ', Δ' of L, we say that the quasi-partition satisfies this sequent iff it is not the case that $\Gamma' \vdash\!\dashv \alpha$, for each $\alpha_1, ..., \alpha_n$ whilst $\Delta' \vdash\!\dashv \beta$, for each $\beta_1, ..., \beta_m$. Now, we have a situation in which the non-normal interpretations are straightforwardly ruled out, primarily because we have symmetry over the turnstile. So, for example, the scenario in which α, $\neg\alpha$ end up in Γ' is ruled out simply by ensuring that we have $\alpha, \neg\alpha \vdash\!\dashv \emptyset$ for all formulas of S. Similarly, the trouble with disjunction is now assuaged since we have $\Gamma, \alpha \vee \beta \vdash\!\dashv \alpha, \beta, \Delta$. The symmetry manifest itself most clearly in the fact that we now have a way of refuting propositions that is symmetrical with the way in which they are proved, and collecting these refuted propositions

[2]For symmetric sequents, this is $\Gamma \vdash\!\dashv \Delta$ iff for some finite $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$ we have $(\Gamma_0 \vdash\!\dashv \Delta_0) \in \mathcal{S}$. See also [8, p.113]. Note that I use $\vdash\!\dashv$ rather than typical $\vdash$ for symmetric sequents to highlight that they can be read in both directions.

together in Δ'.

The trouble with this "fix" is that I do not think it gives us any sort of proof-theoretic semantics at all. It is tricky to interpret symmetric entailment structures, and it is unclear that they provide any account of "proof" whatsoever. According to one prominent interpretation, due to Greg Restall [17], a valid sequent $\Gamma \mid\!\!\mid \Delta$ can be interpreted as saying "it is incoherent to simultaneously assert all of Γ, and deny all of Δ". But, then, we do not have that, for example, (classically) $(\alpha \vee \neg\alpha)$ to be provable, rather it is merely incoherent to deny it. Additionally, according to Lafont [6, Appendix B1], the lack of constructive properties, such as disjunction property, means that "classical logic has no denotational semantics except the trivial one which identifies all the proofs of the same type".[3]

By way of response to the above issue, we might take the restriction on sequents to right-asymmetric form to define intuitionistic, rather than classical, logic. In this setting, the validity of a proof is not dependent upon an interpretation, but rather on local constraints such as canonicity and harmony [15, e.g.]. This way of thinking about valid proofs is not appropriate to classical logic because canonicity requires the disjunction property to hold: a proof of a disjunction must be given in (or reduced to) canonical form, which requires that it is also possible to provide a proof of one of the disjuncts. So, whilst, in a formal sense, symmetric classical logic is stronger than intuitionistic logic (the set of theorems of intuitionistic logic is a proper subset of the theorems of classical logic), informally, classical logic is weaker from the point of view of proofs and the determination of meaning. In symmetric classical logic, we do not have proofs at all. Similarly, we do not have refutations. For instance, say that $\alpha \wedge \beta$ is refuted whenever one of the conjuncts is refuted. But, we can refute $\alpha \wedge \beta$ without having a refutation of either of the conjuncts since the "conjunction" property for refutation fails in classical logic. This is just as problematic from the *p.o.v* of constructing proof-theoretic semantics, since we require an ability to determine, in a fine-grained manner, the manner in which a proof or refutation is "valid".

In [14] it is argued that the problem of categoricity is avoided by this manoeuvre. In this context, the problematic situations would be a scenario in which every statement is provable is compatible with the inferential rules, or it is possible for $\alpha \vee \neg\alpha$ to be provable, whilst neither α nor $\neg\alpha$ are. The reason that these situations can not arise, according to [14], is that there can be no canonical proof of 0 (since there exists only the null introduction rule for 0), so that rules out a scenario in which there exists a proof of both α and $\neg\alpha$ (where $\neg\alpha$ is equivalent to $\alpha \Rightarrow 0$).[4] But, as pointed out in [9], the justification of this rule makes use of the rule for 0-elimination, unlike other rules for connectives, and alternative rules where this is not the case do in fact allow for situations in which proofs for α and $\neg\alpha$ may exist (specifically, where there exists a proof for each atomic formula of the language). The sticking point is that: 'one also needs to regard, controversially, the rule of 0-elimination as justified on the basis of a non-existent rule of 0-introduction, taken as saying that there is no canonical proof of 0' [9]. This is similar to an argument made in [7], where it is pointed out that, whilst the 0-elimination rule only tells us that anything may be inferred from 0, this does not ensure that 0 has the meaning of false. For example, it is possible to consider a language in which all atoms are true, and in which case 0 will be true rather than false, and in which case the 0-elimination rule does not determine the (intended) meaning of 0. Of course, this means that $\neg\alpha$, defined as $\alpha \Rightarrow 0$, must also be true by vacuous discharge, and so, in this language we would have both α and $\neg\alpha$ true.

[3] Note that this result holds for any logic in symmetric form. For example, we may construct a paraconsistent or paracomplete logic in which the left or right negation sequent rule is inadmissible, yet is symmetrical in terms of the structure of sequent derivation (see [17] for further details). But, since Lafont's example does not involve negation, there is no reason why such logics should be capable of dealing with this issue.

[4] Note that I use 0 in place of the usual $\bot$.

So, again, the rules that are supposed to determine the meaning of negation, intuitionistically, do not do so. The issue may be generalised. Typically, when considering the relationship between an entailment structure and a semantic model, counter-models play the role of ruling out invalid formulas. So, for example, take Γ^+ as a set of sentential theorems, V as some model, with $\vdash$ a derivability relation, and $\models$ a model-satisfaction relation. Then, we ordinarily require that, for every formula α: Either $\exists\Gamma^+(\Gamma^+ \vdash \alpha)$ or, $\exists V(V \not\models \alpha)$. That is to appeal to the idea that a sequent $\Gamma \vdash \alpha$ is invalid iff some model V makes all the formulas of Γ true, whilst making α false. It is precisely this appeal to counter-models that is not in keeping with a proof-theoretic approach to semantics, yet it is also what it looks as though to be required given the inability of the asymmetric entailment structures to rule out inadmissible cases. This is perhaps more obvious intuitionistically, since there is a built-in asymmetry between proof (and truth) and refutation (and falsity). For example, truth is directly attainable (by constructing a proof), whilst falsity is typically equated with a reduction to a contradiction, so the latter relies on a syntactic feature (negation), but also on truth. The asymmetry is problematic for the development of a semantics of proofs, where we do not want to rely upon a counter-model as stand-in for refutation. Perhaps what we want instead is something more like this: Either $\exists\Gamma^+(\Gamma^+ \vdash \alpha)$ or, $\exists\Delta^-(\Delta^- \dashv \alpha)$ (where Δ^- is a set of sentential counter-theorems, or refutations, and $\dashv$ a "refutation" entailment relation). Whilst this is possible in symmetrical classical logic, we, arguably, do not get any sort of proofs or refutations in that setting. What we require is some way of maintaining symmetry whilst also ensuring that proofs (and, perhaps refutation) are the primary semantic notions.

2 Co-constructive logic

To maintain logical symmetry over proof and refutation, whilst also ensuring that we have a proof-theoretic semantics that is constructive, I propose to harness the fact that de Morgan duality (without negation) holds between intuitionistic and co-intuitionistic logic, as separate entailment structures. As I have argued elsewhere [18], the latter logic is best understood as a logic of refutation. This will give us a combined entailment structure (co-constructive logic) that is symmetric over proof and disproof (or refutation) without collapsing to classical logic, and which can be interpreted as a dialogue structure between the roles of prover and denier. In general, the idea is that an assertion of a statement brings with it a commitment to its defence, and a denial of the statement involves a commitment to its challenge, with the interaction between the two being constitutive of a proof-theoretic semantics capable of dealing with the above issues. In this setting, de Morgan duality will be shown to bring with it a form of dialogical balance, which is maintained by a kind of cut-rule between prover and denier.

First, take two entaiment structures that formalize proof and refutation. I shall call these L_I for intuitionistic logic, which I take to deal with proofs, and L_C for co-intuitionistic logic, which I take to deal with refutations. First, define two languages S, S^d over a denumerable set of atomic formulas, for L_I and L_C respectively, in Backus-Naur form (α^+ is any atomic formula of S, α^- is any atomic formula of S^d) as:

S $\beta^+ ::= [\alpha^+|(\neg_I\beta^+)|(\beta^+ \wedge \beta^+)|(\beta^+ \vee \beta^+)|(\beta^+ \Rightarrow \beta^+)|0^+]$

S^d $\beta^- ::= [\alpha^-|(\neg_C\beta^-)|(\beta^- \wedge \beta^-)|(\beta^- \vee \beta^-)|(\beta^- \Leftarrow \beta^-)|1^-]$

Here, $\neg_I$ and $\neg_C$ denote the negations of the two languages, and $\Rightarrow$ and $\Leftarrow$ denote implication and co-implication, respectively. These are the key distinctions with classical logic, though as I show below, $\Leftarrow$ essentially operates as a kind of "implication for refutations". Note also that we have made a syntactic distinction between atoms of the two languages, to signify whether they are part of a proof or a refutation. De Morgan duality holds between L_I and L_C, by replacing any use of $\wedge, \vee, \Rightarrow, \neg_I$ in the former, with $\vee, \wedge, \Leftarrow, \neg_C$ in the latter. Also, the dual to a right-asymmetric sequent ($\Gamma \vdash \alpha$) is a left-asymmetric sequent ($\alpha \vdash \Gamma$) [19], though we are reading the latter as a refutation read from right to left, which we indicate by a reverse turnstile, $\dashv$. Whilst the two logical structures are syntactically separated, it is instructive to "see" them within the one and the same proof-theoretic structure.[5] Formulas decorated with $(-)^+$ do not interact with formulas decorated with $(-)^-$, apart from in a rule I call TERMINAL-CUT. Sequents of this, co-constructive, logic, are quadruples, composed of two multisets of formulas, Γ^+ and Δ^-, and two distinguished sets (stoups) containing zero or one formula, Π^+ and Λ^-. This may be noted $\Lambda^-; \Gamma^+ \vdash\!\dashv \Delta^-; \Pi^+$, though ordinarily we will rewrite this as $\alpha^-; \Gamma^+ \vdash\!\dashv \Delta^-; \beta^+$, making transparent that α^- and β^+ are single formulas; where empty, we will write this as $; \Gamma^+ \vdash\!\dashv \Delta^-;$. Superscripts denote as above, and Γ^+ is shorthand for $\alpha_1^+, ..., \alpha_n^+$ for each $\alpha \in \Gamma$ (similarly for Δ^-).[6]

$$\frac{}{;\alpha^+ \vdash\!\dashv ; \alpha^+}\ (\mathrm{Id}^+) \qquad \frac{}{\alpha^-; \vdash\!\dashv \alpha^-;}\ (\mathrm{Id}^-)$$

$$\frac{; \vdash\!\dashv \Delta^-;}{\alpha^-; \vdash\!\dashv \Delta^-;}\ (\text{Thin-}L^-) \qquad \frac{;\Gamma^+ \vdash\!\dashv ; \beta^+}{;\Gamma^+, \alpha^+ \vdash\!\dashv ; \beta^+}\ (\text{Thin-}L^+)$$

$$\frac{\beta^-; \vdash\!\dashv \Delta^-;}{\beta^-; \vdash\!\dashv \Delta^-, \alpha^-;}\ (\text{Thin-}R^-) \qquad \frac{;\Gamma^+ \vdash\!\dashv ;}{;\Gamma^+ \vdash\!\dashv ; \alpha^+}\ (\text{Thin-}R^+)$$

$$\frac{\beta^-; \vdash\!\dashv \Delta^-, \alpha^-, \alpha^-;}{\beta^-; \vdash\!\dashv \Delta^-, \alpha^-;}\ (\mathrm{Cont}^-) \qquad \frac{;\Gamma^+, \alpha^+, \alpha^+ \vdash\!\dashv ; \beta^+}{;\Gamma^+, \alpha^+, \vdash\!\dashv ; \beta^+}\ (\mathrm{Cont}^+)$$

$$\frac{\beta^-; \vdash\!\dashv \Delta^-, \alpha^-, \sigma^-, \Theta^-;}{\beta^-; \vdash\!\dashv \Delta^-, \sigma^-, \alpha^-, \Theta^-;}\ (\mathrm{Int}^-) \qquad \frac{;\Gamma^+, \alpha^+, \sigma^+, \Theta^+ \vdash\!\dashv ; \beta^+}{;\Gamma^+, \sigma^+, \alpha^+, \Theta^+ \vdash\!\dashv ; \beta^+}\ (\mathrm{Int}^+)$$

$$\frac{\beta^-; \vdash\!\dashv \Delta^-, \alpha^-; \qquad \alpha^-; \vdash\!\dashv \Gamma^-;}{\beta^-; \vdash\!\dashv \Delta^-, \Gamma^-;}\ (\mathrm{Cut}^-) \qquad \frac{;\Gamma^+ \vdash\!\dashv ; \alpha^+ \qquad ;\Delta^+, \alpha^+ \vdash\!\dashv ; \beta^+}{;\Delta^+, \Gamma^+ \vdash\!\dashv ; \beta^+}\ (\mathrm{Cut}^+)$$

$$\frac{;\Delta^+, \alpha^+ \vdash\!\dashv ; \sigma^+}{;\Delta^+, \alpha^+ \wedge \beta^+ \vdash\!\dashv ; \sigma^+}\ (\wedge L_1^+) \qquad \frac{;\Delta^+, \beta^+ \vdash\!\dashv ; \sigma^+}{;\Delta^+, \alpha^+ \wedge \beta^+ \vdash\!\dashv ; \sigma^+}\ (\wedge L_2^+)$$

$$\frac{;\Delta^+ \vdash\!\dashv ; \alpha^+ \qquad ;\Delta^+ \vdash\!\dashv ; \beta^+}{;\Delta^+ \vdash\!\dashv ; \alpha^+ \wedge \beta^+}\ (\wedge R^+)$$

$$\frac{\alpha^-; \vdash\!\dashv \Delta^-;}{\alpha^- \wedge \beta^-; \vdash\!\dashv \Delta^-;}\ (\wedge L_1^-) \qquad \frac{\beta^-; \vdash\!\dashv \Delta^-;}{\alpha^- \wedge \beta^-; \vdash\!\dashv \Delta^-;}\ (\wedge L_2^-)$$

$$\frac{\sigma^-; \vdash\!\dashv \Delta^-, \alpha^-; \qquad \sigma^-; \vdash\!\dashv \Delta^-, \beta^-;}{\sigma^-; \vdash\!\dashv \Delta^-, \alpha^- \wedge \beta^-;}\ (\wedge R^-)$$

[5] This is possible by using a technique similar to that used in [5], and in a similar context to this one in [1] (there are also marked differences, particularly regarding the notion of positive and negative rules in those systems).

[6] Note that Cut^+ and Cut^- are elimimable in the combined structure.

$$\frac{;\Delta^+,\alpha^+ \vdash\!\dashv ;\sigma^+ \qquad ;\Delta^+,\beta^+ \vdash\!\dashv ;\sigma^+}{;\Delta^+,\alpha^+ \vee \beta^+ \vdash\!\dashv ;\sigma^+}\,(\vee L^+) \qquad \frac{;\Delta^+ \vdash\!\dashv ;\alpha^+}{;\Delta^+ \vdash\!\dashv ;\alpha^+ \vee \beta^+}\,(\vee R_1^+)$$

$$\frac{;\Delta^+ \vdash\!\dashv ;\beta^+}{;\Delta^+ \vdash\!\dashv ;\alpha^+ \vee \beta^+}\,(\vee R_2^+)$$

$$\frac{\alpha^-; \vdash\!\dashv \Delta^-; \qquad \beta^-; \vdash\!\dashv \Delta^-;}{\alpha^- \vee \beta^-; \vdash\!\dashv \Delta^-;}\,(\vee L^-) \qquad \frac{\sigma^-; \vdash\!\dashv \Delta^-,\alpha^-;}{\sigma^-; \vdash\!\dashv \Delta^-,\alpha^- \vee \beta^-;}\,(\vee R_1^-)$$

$$\frac{\sigma^-; \vdash\!\dashv \Delta^-,\beta^-;}{\sigma^-; \vdash\!\dashv \Delta^-,\alpha^- \vee \beta^-;}\,(\vee R_2^-)$$

$$\frac{;\Delta^+,\alpha^+ \vdash\!\dashv ;\beta^+}{;\Delta^+ \vdash\!\dashv ;\alpha^+ \Rightarrow \beta^+}\,(\Rightarrow R^+) \qquad \frac{;\Delta^+ \vdash\!\dashv ;\alpha^+ \qquad ;\Gamma^+,\beta^+ \vdash\!\dashv ;\sigma^+}{;\Delta^+,\Gamma^+,\alpha^+ \Rightarrow \beta^+ \vdash\!\dashv ;\sigma^+}\,(\Rightarrow L^+)$$

$$\frac{\beta^-; \vdash\!\dashv \Delta^-,\alpha^-;}{\beta^- \Leftarrow \alpha^-; \vdash\!\dashv \Delta^-;}\,(\Leftarrow L^-) \qquad \frac{\sigma^-; \vdash\!\dashv \Delta^-,\alpha^-; \qquad \beta^-; \vdash\!\dashv \Gamma^-;}{\sigma^-; \vdash\!\dashv \Delta^-,\Gamma^-,\alpha^- \Leftarrow \beta^-;}\,(\Leftarrow R^-)$$

$$\frac{;\Delta^+,\alpha^+ \vdash\!\dashv ;}{;\Delta^+ \vdash\!\dashv \neg_I \alpha^+;}\,(\neg_I R^+) \qquad \frac{;\Delta^+ \vdash\!\dashv ;\alpha^+}{;\Delta^+,\neg_I \alpha^+ \vdash\!\dashv ;}\,(\neg_I L^+) \qquad \frac{; \vdash\!\dashv \Delta^-,\alpha^-;}{\neg_C \alpha^-; \vdash\!\dashv \Delta^-;}\,(\neg_C R^-)$$

2.1 Interpreting co-constructive logic

In brief, we will consider a dialogue in the above structure in terms of a relationship between PROOF ATTEMPTS and REFUTATION ATTEMPTS that are "tests" of each other. As such, the turnstile, $\vdash\!\dashv$ may be read positively, from left to right, as a proof-attempt, and negatively, from right to left, as a refutation attempt. For example, the rules $\wedge L_1^-$ and $\wedge L_2^-$ may together be interpreted in BHK-style as "c is a refutation attempt of $\alpha \wedge \beta$ if c is a pair (l, c') such that c' is a refutation attempt of α or c is a pair (r, c') such that c' is a refutation attempt of β". This coheres with a view presented in [10], which suggests that making an assertion involves a commitment to defend it when challenged. So, assertoric norms should not be understood to restrict what an agent ought to assert, instead they are constraints on how agents respond to challenge and dialogue. This is related to inferentialism in [11], where it is argued that "an act of asserting a statement brings with it a commitment to defend the assertion, if challenged, so to make an assertion is to make a move in a game [...] the 'game of asking for and giving reasons' is embedded in the very nature of assertions". Here, we will say that an interaction consists of an assertion of a statement, which brings with it a commitment to its defence, or a denial of the statement, which involves a commitment to its challenge. It is this interaction that we will say provides an interpretation of the statement through its testing. For example, a "test" for a proof-attempt of a conjunction is disjunctive, so is more fine-grained than any counter-model since we test each formula used as a premise for the proof-attempt.

Take the following interaction. "Prover" asserts a conjunction, $\alpha \wedge \beta$, putting $\alpha \wedge \beta$ "into the game", and it is incumbent upon prover, to provide a proof attempt giving some sort of reason or evidence in support of both α and β. Refuter, on the other hand, challenges $\alpha \wedge \beta$ by providing a refutation attempt giving some sort of reason or evidence refuting either α or β. Exactly the reverse is the case if the formula in question is a disjunction. So, we can interpret the relationship between L_I and L_C in terms of tests, where a test of α^+ is just a refutation-attempt of form α^-, and a test of α^- is just a proof-attempt of form α^+: Testing $\alpha^+ \wedge \beta^+$

involves testing α^+ or testing β^+; Testing $\alpha^+ \vee \beta^+$ involves testing α^+ and testing β^+; Testing $\alpha^- \wedge \beta^-$ involves testing α^- and testing β^-; Testing $\alpha^- \vee \beta^-$ involves testing α^- or testing β^-; Testing $\alpha^+ \Rightarrow \beta^+$ involves testing for a function that maps each test of α^+ into a test of β^+; Testing $\beta^- \Leftarrow \alpha^-$ involves testing for a function that maps each test of α^- into a test of β^-.

Whereas this is fairly intuitive for conjunction and disjunction, that a "test" of an attempted proof of $\alpha \Rightarrow \beta$ is an attempted refutation of $\beta \Leftarrow \alpha$ is less obvious. However, understanding why this is the case also sheds light on the more general issue that we also introduced here the idea that proofs (and refutations) may not be valid, since proof-attempts are open to testing, counter-examples, and challenge. Note, first, that this rules out a conception of proofs as tenseless objects. Rather, following [4], we will take the view that proofs (and refutations) are acts: "a proof is a process whose result may be represented or described by means of linguistic symbols". According to [12], this alters the standard intuitionistic definition of conditional so that $\alpha \Rightarrow \beta$ may be valid only in case an agent has an actual proof of α to hand. A proof of a conditional is a function that maps actual proofs of the antecedent into actual proofs of the consequent since: 'as long as no proof of α is known, [the function] f has nothing to map. So we can still define f as the constant function which, once a proof π of α is known, maps every proof of α into the proof of β'.[7] So, a valid proof of a conditional $\alpha \Rightarrow \beta$ requires a proof of α in addition to a proof that there exists a function mapping every proof of α into a proof of β.

This can be usefully mapped on to a distinction between proof, or refutation, *attempts*, and *valid* proofs and refutations. Consider this in terms of the detachable forms of the entailment relations $\alpha \vdash \beta$ and $\alpha \dashv \beta$. Intuitionistically, implication is a detachable operator that embeds provability such that $\alpha \vdash_I \beta \leftrightarrow \vdash_I \alpha \Rightarrow \beta$. In co-intuitionistic logic, co-implication is a detachable operator that embeds refutability such that $\alpha \dashv_C \beta \leftrightarrow \dashv_C \beta \Leftarrow \alpha$.[8] So, by deduction (and dual-deduction) theorem, we can read a sequent $\Gamma \vdash \alpha$ as "under the assumption that there exists a proof of each $\alpha \in \Gamma$, there exists a proof of β also" (similarly for refutations and $\Gamma \dashv \alpha$). This is a reading in the form of a conditional, and so deserves to come under the rubric of proof-attempt. A valid proof, according to this distinction, requires, in addition, that we can also provide proofs for each $\alpha \in \Gamma$.[9] A proof-attempt is analogous with a conditional for which a function does not yet exist since we do not know if there is anything for it to map; a valid proof is one in which deduction theorem holds since we have valid proofs for each premise (so they can be introduced as axioms), so the conclusion may be taken as a theorem. Co-implication is symmetrical. This way of considering implication and co-implication makes things more transparent since we can not detach on "both sides" as it were: a formula α can not have a valid proof and a valid refutation (simultaneously at least). So, whilst we can consider the hypothetical proof and refutation of implication and co-implication in one and the same moment, only one can "win".

Negation is altered by this approach since the distinction between proof-attempts and valid proofs is at odds with intuitionistic negation [13]. Simply put, the negation of α can never be valid if it is defined as $\alpha \Rightarrow 0$, whilst also requiring that in order for the conditional to be valid, there must also exist a proof of α. Whilst this is an issue for intuitionism, it provides grist to the mill for the account offered here, in which there must be a balance between negative and positive

[7]There is an obvious analogy to Ramsey's [16] argument that to assert a conditional is not asserting a conditional proposition, but to make a conditional assertion of the consequent.

[8]Note that one of the reasons for syntactic separation is due to the fact that L_I can not be extended by co-implication, nor can L_C be extended by implication, and any attempt to write them in by definition collapses to classical logic.

[9]Prawitz [15] makes a similar distinction between open and closed arguments, where open arguments involve undischarged assumptions, or unbound variables, whilst closed arguments contain no assumptions, and are valid just in case it is either a canonical argument (i.e. an argument that ends with an instance of an introduction rule), or it can be reduced to a canonical argument for the conclusion.

logics, and the role played by $\neg\alpha$ in constructive logic is played by $\dashv \alpha^-$. This also means that the negation defined via implication is an expressive advance on the usual interpretation of intuitionistic negation, which enables us to simply say that a proof (or refutation)-attempt *does not go through*.

2.2 Dialogical coherence and termination

This brings us to the relationship between a valid proof or refutation, and the termination of dialogue. In addition to the above rules, we have a rule that is admissible under certain circumstances, which we call TERMINAL-CUT.[10] First, define counterpart formulas between the two logics as follows: for any formula α with superscript $^{+,-}$, the counterpart formula, denoted $\alpha_\circ$, is simply the same formula with the opposite superscript, apart from $\Rightarrow$, whose counterpart is $\Leftarrow$.[11] So, for example, the counterpart of $\alpha^+ \wedge \beta^+$ is just $\alpha^- \wedge \beta^-$. With this, we define terminal-cut as follows.

$$\frac{\alpha_\circ^-; \Gamma^+ \vDash\!\dashv \Delta^-; \qquad ; \Gamma^+ \vDash\!\dashv \Delta^-; \alpha_\circ^+}{; \Gamma^+ \vDash\!\dashv \Delta^-;} \text{ (Terminal-cut)}$$

Terminal cut is admissible for dialogues that terminate with either prover or denier "winning", and may be understood as inducing a kind of harmony between the two logics. Harmony ordinarily operates across introduction and elimination rules inside the same logical structure, the idea being that there must be a balance between the grounds for the possible assertion of a formula with the consequences of accepting it.[12] Here, there is also a form of balance between positive and negative formulas of the logics by the fact that rules for counterpart formulas are related by de Morgan duality, and so are secondary to this balance between proof- and refutation-attempts over the course of an interaction. We define a dialogue as COHERENT iff it is the case that *if terminal-cut were to be applied*, then either: empty sequent would be "derived" in the course of a dialogue; or a canonical introduction of a proof or refutation is derived if the dialogue is terminated (in both cases we should add the clause that the proof or refutation can be reduced to a canonical proof or refutation if it is not already). Stipulating that terminal-cut must be admissible requires that a formula with a dominant operator can not be canonically introduced as both a proof and a refutation *at the termination of dialogue*.

With this in hand, we can provide a definition of validity that ensures the semantic role of proofs and refutations in a localised manner (i.e. without reference to idealised or global constraints upon soundness and completeness): Say that a proof or refutation is VALID iff: (a) It is closed; (b) It is canonical, or can be reduced to a canonical proof or refutation; (c) It is the result of a termination of a coherent dialogical interaction such that terminal-cut is admissible (without removing the formula with dominant operator). Whilst (a) and (b) are familiar from the literature mentioned above, (c) reflects the fact that we are concerned with an interaction between hypothetical proofs and tests, such that a valid formula is one that is introduced at the *end* of that interaction that terminates with an agreement between agents involved in the dialogue. The semantic role played by a valid proof or refutation is, therefore, dependent

[10] We need to be careful about calling this a rule, since it really just formalises termination of dialogue.

[11] The reason for this is simply due to notation since the same role is being played for proofs and $\Rightarrow$, and refutations and $\Leftarrow$

[12] Though, because we see proofs as acts, unlike in [15], harmony follows from the prior requirement that cut is eliminable. For example, we know that, for any proof of $\alpha \wedge \beta$, it must be possible to extract proofs of α, and proofs of β, since if this were not the case, then it would not be possible to form a cut-free proof with that conclusion. Cut-elimination is, therefore, a key requirement in defining the validity of proofs, which, of course, ensures the transitivity of deduction so that validity is preserved.

upon the success of either, but not both. It is not the case that rules of inference determine meaning by themselves, but rather that the correctness of these rules is already dependent upon dialogical coherence. Sloganised, the meaning of a proposition is built-up through this process of interaction between proof and refutation attempts.

As is obvious, problems of categoricity and compositionality do not arise for valid formulas, since it is no longer possible for both α^+ and α^- to be valid, and neither is it possible for $(\alpha^+ \vee \alpha^-)$ to be valid whilst neither α^+ nor α^- is (for any formula, α). Furthermore, far from requiring access to idealised Lindenbaum chains, or semantic models, a local completeness theorem arises naturally from the constraints on the validity of proofs and refutations given above: Let $|P|$ denote a terminating dialogue ending with a valid proof, $|R|$ a terminating dialogue ending with a valid refutation, and α^+, α^-, denote proof and refutation attempts of a formula α, respectively. Then, for any α^+, α^- whose interaction has terminated there exists either a valid proof P of α^+ such that $P = |P|$, or a valid refutation R of α^- such that $R = |R|$. Then, $|P|$ or $|R|$, are said to interpret the formula α. It is important to note that this theorem refers only to interactions that have terminated, it does not require that each interaction will terminate, and is localised in that it makes no reference to proof (and refutation) attempts that are outside the current context of the dialogue in question. For example, take the case of a dialogue involving a proof-attempt of $\alpha \wedge \beta$, and with an attempted refutation through α (the notation $[A_1]$ just indicates subproofs (or subrefutations)):

$$\dfrac{\dfrac{\begin{matrix}[A_1]\\ ;\Gamma^+ \vdash\dashv ;\alpha^+\end{matrix} \qquad \begin{matrix}[A_2]\\ ;\Gamma'^+ \vdash\dashv ;\beta^+\end{matrix}}{;\Gamma^+,\Gamma'^+ \vdash\dashv ;\alpha^+ \wedge \beta^+}(\wedge R^+) \qquad \dfrac{\begin{matrix}[B]\\ \alpha^-; \vdash\dashv \Delta^-;\end{matrix}}{\alpha^- \wedge \beta^-; \vdash\dashv \Delta^-;}(\wedge L_1^-)}{;\Gamma^+,\Gamma^{+\prime} \vdash\dashv \Delta^-;}\text{(Terminal-cut)}$$

Equally, "denier" could attempt to refute via β, with the same result. As is obvious, this also means that the dominant formulas can be eliminated through the usual process of cut-elimination by pushing cuts upwards. For example, for α:

$$\dfrac{\begin{matrix}[A_1]\\ ;\Gamma^+ \vdash\dashv ;\alpha^+\end{matrix} \qquad \begin{matrix}[B]\\ \alpha^-; \vdash\dashv \Delta^-;\end{matrix}}{;\Gamma^+ \vdash\dashv \Delta^-;}\text{(Terminal-cut)}$$

What about the case in which, say, the refutation-attempt is valid? This requires that the subrefutation $[B]$ of all $\sigma \in \Delta^+$ is valid, which we denote just by $[B_V]$. Then, the case where there exists a valid refutation of $\alpha \wedge \beta$ looks like this:

$$\dfrac{\dfrac{\begin{matrix}[A_1]\\ ;\Gamma^+ \vdash\dashv ;\alpha^+\end{matrix} \qquad \begin{matrix}[A_2]\\ ;\Gamma'^+ \vdash\dashv ;\beta^+\end{matrix}}{;\Gamma^+,\Gamma'^+ \vdash\dashv ;\cancel{\alpha^+ \wedge \beta^+}}(\wedge R^+) \qquad \dfrac{\begin{matrix}[B_V]\\ \alpha^-; \vdash\dashv \Delta^-;\end{matrix}}{\alpha^- \wedge \beta^-; \vdash\dashv \Delta^-;}(\wedge L_1^-)}{\alpha^- \wedge \beta^-;\Gamma^+,\Gamma^{+\prime} \vdash\dashv \Delta^-;}\text{(Terminal-cut)}$$

Where we use $\cancel{\alpha^+ \wedge \beta^+}$ to indicate that the formula $\alpha^+ \wedge \beta^+$ is not introduced due to the existence of a valid refutation of α^- so the subproof attempt, $[A_1]$, of α^+ fails, and there can be no introduction of $\alpha^+ \wedge \beta^+$. Then, the above refutation R can subsequently be written just like this:

$$\dfrac{\begin{matrix}[B_V] \\ \alpha^-; \vdash\!\dashv \Delta^-;\end{matrix}}{\alpha^- \wedge \beta^-; \vdash\!\dashv \Delta^-;}\ (\wedge L_1^-)$$

Moreover, this valid refutation R is equivalent with the version that went via a terminating dialogue, so $R = |R|$ holds.

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