Epistemic Modals, Qualitative Probability, and Nonstandard Probability

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Abstract

Yalcin [23] shows that Kratzer's model [5] does not validate some intuitively valid inferences and validates some intuitively invalid ones. He adopts a model based directly on a probability measure. However, as Kratzer [6] says, 'Our semantic knowledge alone does not give us the precise quantitative notions of probability and desirability that mathematicians and scientists work with', Yalcin's model seems to be unnatural as a model for comparative epistemic modals. The aim of this paper is to propose a new version of complete logic—modal-qualitative-probability logic (MQPL)—the model of the language of which has the following four merits: (i) The model reflects Kratzer's intuition above in the sense that the model is not based directly on a probability measure, but based on a qualitative probability ordering. (ii) The model does not cause Yalcin's problem. (iii) The model has no limitation of the size of the domain. (iv) The model can deal with the two-dimensional geometric probability that Kolmogorov probability theory cannot.

1 Motivation

Kratzer [5] provides comparative epistemic modals such as 'at least as likely as' with their models in terms of a qualitative ordering on propositions derived from a qualitative ordering on possible worlds. Yalcin [23] shows that Kratzer's model does not validate some intuitively valid inference schemata and validates some intuitively invalid ones. He adopts a model based directly on a probability measure for comparative epistemic modals. His model does not cause this problem. However, as Kratzer [6] says, 'Our semantic knowledge alone does not give us the precise quantitative notions of probability and desirability that mathematicians and scientists work with', Yalcin's model seems to be unnatural as a model for comparative epistemic modals. Holliday and Icard [3] prove that not only a probability measure model but also a qualitatively additive measure model and a revised version of Kratzer's model do not cause Yalcin's problem.

The aim of this paper is to propose a new version of complete logic—modal-qualitative-probability logic (MQPL)—the model of the language of which has the following four merits:

1. The model reflects *Kratzer's intuition* above in the sense that the model is not based directly on a probability measure, but based on a *qualitative probability ordering*.
2. The model does not cause Yalcin's problem.
3. The model has *no limitation of the size* of the domain.
4. The model can deal with the *two-dimensional geometric probability*, for example, of picking a point from the diagonals D_1 and D_2 of a rectangle, given that the point is on one of D_1 and D_2, equals $\frac{1}{2}$ under *nonstandard infinitesimal* probability theory and equals $\frac{0}{0}$ (*undefined*) under *Kolmogorov* probability theory. In other words, we can provide the

following sentence with its truth condition in terms of the model of the language of MQPL, but cannot provide in terms of Kolmogorov probability measure model:

The point being on D_1 is as likely as it being on D_2.

So the model of the language of MQPL has a wider scope than Kolmogorov probability measure model.

The structure of this paper is as follows. In Section 2, we argue on some relations between qualitative probability and a standard probability measure and a relation between qualitative probability and a nonstandard probability measure. In Section 3, we define the language $\mathcal{L}_{\mathsf{MQPL}}$ of MQPL, define a model $\mathfrak{M}$ of $\mathcal{L}_{\mathsf{MQPL}}$, provide MQPL with a truth definition and a validity definition, show that MQPL justifies the (in)validity of Yalcin's formulae, provide MQPL with its proof system, and touch upon the soundness and completeness theorems of MQPL. In Section 4, we compare MQPL with some logics of Holliday and Icard [3]. In Section 5, we finish with brief concluding remarks.

2 Qualitative Probability and Nonstandard Probability

When W is a nonempty set of possible worlds, $\mathcal{F}$ a Boolean algebra of subsets of W, and $\succsim$ a qualitative probability ordering on $\mathcal{F}$, de Finetti [1] specifies necessary conditions on $\langle W, \mathcal{F}, \succsim \rangle$ for the existence of a probability measure. Kraft, Pratt, and Seidenberg [4] shows that de Finetti's conditions on $\langle W, \mathcal{F}, \succsim \rangle$ are not sufficient and presents necessary and sufficient conditions on $\langle W, \mathcal{F}, \succsim \rangle$ for the existence of a probability measure in the case that W is *finite*. Scott [8] presents much the same conditions in a general setting as follows:

Definition 1 (Nontriviality, Nonnegativity, and Scottness (S))**.**

- ***Nontriviality****:* $W \succ \emptyset$.
- ***Nonnegativity****:* $A \succsim \emptyset$ *for any* $A \in \mathcal{F}$.
- ***Connectedness****:* $A \succsim B$ *or* $B \succsim A$ *for any* $A, B \in \mathcal{F}$.
- ***Scottness (S)****: For any* $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{F}$*, if, for any* $i < n$*,* $(A_i \succsim B_i)$*, then* $B_n \succsim A_n$*, given that*

$$\sum_{i=1}^{n} \chi_{A_i} = \sum_{i=1}^{n} \chi_{B_i}$$

holds, where χ_A *is a characteristic function of* $A \in \mathcal{F}$*.* χ_A *is a function from* W *to* $\{0, 1\}$ *such that*

$$\chi_A(w) := \begin{cases} 1 & \text{if} \quad w \in A, \\ 0 & \text{if} \quad w \notin A. \end{cases}$$

Remark 1. *Intuitively,* ***Scottness (S)*** *says that whenever, for any* $w \in W$*,* w *is in* ***exactly as many*** A_i*'s* ***as*** B_i*'s, if, for any* $i < n$*,* $(A_i \succsim B_i)$*, then* $B_n \succsim A_n$*.*

Narens [7] shows that the same same conditions as Scott are necessary and sufficient for the existence of a *nonstandard* probability measure *without the limitation of the size* of W. However, **Scottness (S)** is unpleasant because it is stated in terms of *characteristic functions* rather than in terms of union or other primitive notions. Domotor [2] states the condition in terms of intersection, union, and $\succsim$ as follows:

Definition 2 (Scottness (D))**.** ***Scottness (D)**: For any $A_1, \dots, A_n, B_1, \dots, B_n \in \mathcal{F}$, if for any $i < n$, ($A_i \succsim B_i$), then $B_n \succsim A_n$, given that*

$$\bigcup_{1 \leq i_1 < \cdots < i_k \leq n} (A_{i_1} \cap \cdots \cap A_{i_k}) = \bigcup_{1 \leq i_1 < \cdots < i_k \leq n} (B_{i_1} \cap \cdots \cap B_{i_k})$$

holds for any k with $1 \leq k \leq n$.

When ${}^*\mathbb{R}$ denotes the set of *nonstandard reals* (containing *infinitesimals*), we can state *Scott-Narens theorem* by using *Domotor's notation* as follows:

Theorem 1 (Representation for Qualitative Probability Ordering)**.** *For any $\langle W, \mathcal{F}, \succsim \rangle$, there exists a finitely additive **nonstandard**-real-valued probability measure $P : \mathcal{F} \to^* \mathbb{R}$ satisfying*

$$A \succsim B \quad \textit{iff} \quad P(A) \geq P(B)$$

*iff **Nontriviality**, **Nonnegativity**, **Connectedness**, and **Scottness (D)** are met.*

3 Modal-Qualitative-Probability Logic (MQPL)

3.1 Language

We define the language $\mathcal{L}_{\mathsf{MQPL}}$ of MQPL, which is the *same language as Holliday and Icard [3]*, as follows:

Definition 3 (Language)**.** *Let $\mathcal{S}$ denote a set of sentential variables, $\Diamond$ a unary sentential operator, and $\geqslant$ a binary sentential operator. The language $\mathcal{L}_{\mathsf{MQPL}}$ of MQPL is given by the following BNF grammar:*

$$\varphi ::= s \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid (\varphi \geqslant \varphi)$$

such that $s \in \mathcal{S}$.

- $\bot, \vee, \to, \leftrightarrow$ *and* $\Box$ *are introduced by the standard definitions.*
- $\Diamond\varphi$ *is interpreted to mean that **possibly** φ.*
- $\varphi \geqslant \psi$ *is interpreted to mean that φ is **at least likely as** ψ.*
- $\varphi > \psi := (\varphi \geqslant \psi) \wedge \neg(\psi \geqslant \varphi)$.
- $\varphi > \psi$ *is interpreted to mean that φ is **more likely than** ψ.*
- $\triangle\varphi := \varphi > \neg\varphi$.
- $\triangle\varphi$ *is interpreted to mean that **probably** φ*
- *The set of all well-formed formulae of $\mathcal{L}_{\mathsf{MQPL}}$ will be denoted by $\Phi_{\mathcal{L}_{\mathsf{MQPL}}}$.*

3.2 Semantics

We define a structured model $\mathfrak{M}$ of $\mathcal{L}_{\mathsf{MQPL}}$ as follows:

Definition 4 (Model)**.** *$\mathfrak{M}$ is a quadruple $\langle W, R, V, \rho \rangle$ in which*

- *W is a non-empty set of possible worlds,*
- *R is a binary accessibility relation on W,*
- *V is a truth assignment to each $s \in \mathcal{S}$ for each $w \in W$, and*
- *ρ is a qualitative probability space assignment that assigns to each $w \in W$ a qualitative probability space $\langle W_w, \mathcal{F}_w, \succsim_w \rangle$ in which*
  - *$W_w := \{w' \in W : R(w, w')\}$,*
  - *$\mathcal{F}_w$ is a Boolean algebra of subsets of W_w with $\emptyset$ as zero element and W_w as unit element, and*
  - *$\succsim_w$ is a qualitative probability ordering on $\mathcal{F}_w$ that satisfies all of **Nontriviality**, **Nonnegativity**, **Connectedness**, and **Scottness (D)** of Theorem 1.*

We provide MQPL with the following truth definition at $w \in W$ in $\mathfrak{M}$, define the truth in $\mathfrak{M}$, and then define validity as follows:

Definition 5 (Truth and Validity)**.** *The notion of $\varphi \in \Phi_{\mathcal{L}_{\mathsf{MQPL}}}$ being true at $w \in W$ in $\mathfrak{M}$, in symbols $(\mathfrak{M}, w) \models_{\mathsf{MQPL}} \varphi$, is inductively defined as follows:*

- $(\mathfrak{M}, w) \models_{\mathsf{MQPL}} s$ *iff* $V(w)(s) = \mathbf{true}$.
- $(\mathfrak{M}, w) \models_{\mathsf{MQPL}} \top$.
- $(\mathfrak{M}, w) \models_{\mathsf{MQPL}} \neg\varphi$ *iff* $(\mathfrak{M}, w) \not\models_{\mathsf{MQPL}} \varphi$.
- $(\mathfrak{M}, w) \models_{\mathsf{MQPL}} \varphi \wedge \psi$ *iff* $(\mathfrak{M}, w) \models_{\mathsf{MQPL}} \varphi$ *and* $(\mathfrak{M}, w) \models_{\mathsf{MQPL}} \psi$.
- $(\mathfrak{M}, w) \models_{\mathsf{MQPL}} \Diamond\varphi$ *iff, for some w' such that $R(w, w')$,* $(\mathfrak{M}, w') \models_{\mathsf{MQPL}} \varphi$.
- $(\mathfrak{M}, w) \models_{\mathsf{MQPL}} \varphi \geqslant \psi$ *iff* $[\![\varphi]\!]_w^{\mathfrak{M}} \succsim_w [\![\psi]\!]_w^{\mathfrak{M}}$, *where* $[\![\varphi]\!]_w^{\mathfrak{M}} := \{w' \in W : R(w, w')$ *and* $(\mathfrak{M}, w') \models_{\mathsf{MQPL}} \varphi\}$.

If $(\mathfrak{M}, w) \models_{\mathcal{L}_{\mathsf{MQPL}}} \varphi$ for all $w \in W$, we write $\mathfrak{M} \models_{\mathcal{L}_{\mathsf{MQPL}}} \varphi$ and say that φ is true in $\mathfrak{M}$. If φ is true in all models of $\mathcal{L}_{\mathsf{MQPL}}$, we write $\models_{\mathcal{L}_{\mathsf{MQPL}}} \varphi$ and say that φ is valid.

The next corollary follows from Theorem 1 and Definition 5.

Corollary 1 (Truth Condition by Probability Measure)**.** *There exists a finitely additive nonstandard-real-valued probability measure $P : \mathcal{F} \to^* \mathbb{R}$ satisfying*

$$(\mathfrak{M}, w) \models_{\mathsf{MQPL}} \varphi \geqslant \psi \quad \textit{iff} \quad P([\![\varphi]\!]_w^{\mathfrak{M}}) \geq P([\![\psi]\!]_w^{\mathfrak{M}}).$$

*iff **Nontriviality**, **Nonnegativity**, **Connectedness**, and **Scottness (D)** are met.*

Yalcin (2010) presents the following list of intuitively valid formulae (V1)–(V11) and intuitively invalid formulae (I1) and (I2):

- (V1) $\triangle\varphi \to \neg\triangle\neg\varphi$,

- (V2) $\triangle(\varphi \wedge \psi) \to (\triangle\varphi \wedge \triangle\psi)$,
- (V3) $\triangle\varphi \to \triangle(\varphi \vee \psi)$,
- (V4) $\varphi \geqslant \bot$,
- (V5) $\top \geqslant \varphi$,
- (V6) $\Box\varphi \to \triangle\varphi$,
- (V7) $\triangle\varphi \to \Diamond\varphi$,
- (V8) $(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$,
- (V9) $(\varphi \to \psi) \to (\neg\triangle\psi \to \neg\triangle\varphi)$,
- (V10) $(\varphi \to \psi) \to (\psi \geqslant \varphi)$,
- (V11) $(\psi \geqslant \varphi) \to (\triangle\varphi \to \triangle\psi)$,
- (V12) $(\psi \geqslant \varphi) \to ((\varphi \geqslant \neg\varphi) \to (\psi \geqslant \neg\psi))$,
- (I1) $((\varphi \geqslant \psi) \wedge (\varphi \geqslant \chi)) \to (\varphi \geqslant (\psi \vee \chi))$, and
- (I2) $((\varphi \geqslant \neg\varphi) \wedge (\neg\varphi \geqslant \varphi)) \to (\varphi \geqslant \psi)$.

MQPL justifies the (in)validity of Yalcin's formulae as follows:

Proposition 1 (Justification of Yalcin's Formulae)**.** MQPL *validates all of (V1)–(V12) and validate neither (I1) nor (I2).*

3.3 Syntax

The proof system of MQPL consists of the following:

Definition 6 (Proof System)**.**

- *All tautologies of classical sentential logic,*
- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ *(K)*,
- $(\Box(\varphi_1 \leftrightarrow \varphi_2) \wedge \Box(\psi_1 \leftrightarrow \psi_2)) \to ((\varphi_1 \geqslant \psi_1) \leftrightarrow (\varphi_2 \geqslant \psi_2))$
  (Replacement of Necessary Equivalents),
- $\top > \bot$ *(Syntactic Counterpart of **Nontriviality**),*
- $\varphi \geqslant \top$ *(Syntactic Counterpart of **Nonnegativity**),*
- $(\varphi \geqslant \psi) \vee (\psi \geqslant \varphi)$ *(Syntactic Counterpart of **Connectedness**),*
- $$\left(\bigvee_{1 \leq i_1 < \cdots < i_k \leq n} (\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}) \leftrightarrow \bigvee_{1 \leq i_1 < \cdots < i_k \leq n} (\psi_{i_1} \wedge \cdots \wedge \psi_{i_k})\right) \to \left(\bigwedge_{i=1}^{n-1} (\varphi_i \geqslant \psi_i) \to (\psi_n \geqslant \varphi_n)\right)$$
  *(Syntactic Counterpart of **Scottness (D)**),*

- *Modus Ponens, and*
- *Necessitation.*

A proof of $\varphi \in \Phi_{\mathcal{L}_{\mathsf{MQPL}}}$ is a finite sequence of $\mathcal{L}_{\mathsf{MQPL}}$-formulae having φ as the last formula such that either each formula is an instance of an axiom or it can be obtained from formulae that appear earlier in the sequence by applying an inference rule. If there is a proof of φ, we write $\vdash_{\mathsf{MQPL}} \varphi$.

3.4 Metalogic

We can prove the soundness and completeness of MQPL.

Theorem 2 (Soundness)**.** *For any $\varphi \in \Phi_{\mathcal{L}_{\mathsf{MQPL}}}$, if $\vdash_{\mathsf{MQPL}} \varphi$, then $\models_{\mathcal{L}_{\mathsf{MQPL}}} \varphi$.*

Theorem 3 (Completeness)**.** *For any $\varphi \in \Phi_{\mathcal{L}_{\mathsf{MQPL}}}$, if $\models_{\mathcal{L}_{\mathsf{MQPL}}} \varphi$, then $\vdash_{\mathsf{MQPL}} \varphi$.*

4 Comparison with Holliday and Icard [3]

In this section, we would like to compare MQPL with some logics of Holliday and Icard [3]. We have adopted the same language $\mathcal{L}_{\mathsf{MQPL}}$ as $\mathcal{L}$ of [3]. Holliday and Icard consider three kinds of models for $\mathcal{L}$:

1. measure model,
   (a) finitely additive measure model (finitely additive probability measure model),
   (b) qualitatively additive measure model,
2. event-ordering model,
3. world-ordering model
   (a) Kratzer's world-ordering model,
   (b) revised version of Kratzer's world-ordering model.

Holliday and Icard show that a complete logic **WJR** based on Kratzer's world-ordering model validate all of (V1)–(V10), (V12), (I1), and (I2), but does not validate (V11), and that a complete logic $\mathbf{FP}_\infty$ based on a finitely additive measure model, a complete logic **FA** based on a qualitatively additive measure model, and a logic $\mathbf{WP}_\infty\mathbf{R}$ based on a revised version of Kratzer's world-ordering model each validate all of (V1)–(V12) and validate neither (I1) nor (I2) On the other hand, we have proposed a complete logic MQPL based on a kind of *event-ordering* model—-*qualitative probability ordering* model. This model is a compromise between an event-ordering model and a finitely additive measure model in the sense that by *Theorem 1* (Representation for Qualitative Probability Ordering), the event-ordering model (qualitative probability ordering model) can be connected to the finitely additive measure model (finitely additive nonstandard-real-valued probability measure model) in *Corollary 1* (Truth Condition by Probability Measure). The qualitative probability ordering side of the model $\mathfrak{M}$ of $\mathcal{L}_{\mathsf{MQPL}}$ can reflect Kratzer's intuition above in the sense that the model is not based directly on a probability measure, but based on a qualitative probability ordering. The finitely additive nonstandard-real-valued probability measure side of $\mathfrak{M}$ enables MQPL to avoid Yalcin's problem.

5 Concluding Remarks

In this paper, we have proposed a new version of complete logic—modal-qualitative-probability logic (MQPL)—the model of the language of which has the four merits listed in Section 1.

This paper is only a part of a larger *measurement-theoretic* study. By means of measurement theory, we constructed or are trying to construct such logics as

1. (dynamic epistemic) preference logic [9, 11],
2. dyadic deontic logic [10],
3. vague predicate logic [14, 15]
4. threshold-utility-maximiser's preference logic [12, 13],
5. interadjective-comparison logic [18],
6. gradable-predicate logic [17],
7. logic for better questions and answers [16],
8. doxastic and epistemic logic [22],
9. multidimensional-predicate-comparison logic [19],
10. logic for preference aggregation represented by a Nash collective utility function [20], and
11. preference aggregation logic for weighted utilitarianism [21].

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