PROOF-THEORETIC SEMANTICS FOR A SYLLOGISTIC FRAGMENT

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1. Introduction

We present some *prolegomena* to *Proof-Theoretic Semantics (PTS)* for natural language (NL). The following quotation from Schroeder-Heister 2005 emphasizes the lack of applicability to NL, the original reason for PTS to start with:

> Although the *"meaning as use"* approach has been quite prominent for half a century now and provided one of the cornerstones of philosophy of language, in particular of ordinary language philosophy, it has never become prevailing in the *formal* semantics of artificial and natural languages. In formal semantics, the *denotational* approach which starts with interpretations of singular terms and predicates, then fixes the meaning of sentences in terms of truth conditions, and finally defines logical consequence as truth preservation under all interpretations, has always dominated.

In order to device a PTS for (a fragment of) NL, two steps are required:
1. Device a *proof-theory (a calculus)* for the fragment, satisfying criteria proposed for PTS in logic. Replace *truth condition* by *derivability conditions* (in the above calculus) as the meaning of sentences in the fragment.
2. Identify the contribution of subsentential phrases (down to words) to the PTS meaning of sentences in which they occur.
Here, we focus on the first task only.

The studied fragment is SYL (syllogistic logic) Moss 2005, where Moss considers its Hilbert-like axiomatization, being concerned mainly with completeness w.r.t. set-based "natural" semantics, and extensions not expressible in 1st-order logic. The fragment is: $All\ X\ are\ Y \quad Some\ X\ are\ Y \quad No\ X\ are\ Y \quad J\ is\ an\ X$
X, Y range over predicate symbols, and J as an individual constant. Here we only study the *positive* fragment SYL^+, without $No\ X\ are\ Y$.

We propose a *natural deduction* proof system for SYL, with proof-terms embodying a *Curry-Howard (CH)* correspondence. The system is shown to be *harmonious*, taken here as the requirement that its rules satisfy *local soundness (LS)* and *local completeness (LC)* Pfenning and Davies 2001. LS requires that *every* introduction immediately followed by elimination is *reducible* to a derivation without such *detour*. Failing LS means elimination is too strong. LC requires that for every elimination *there is* a reconstructing introduction. Failing LC means elimination is too weak.

The proof-terms are *drawn from* the traditional λ-calculus, but receive a somewhat different interpretation via a *BHK*-like justification of the deduction-rules. All rules in Moss 2005 are derivable in our system, rendering it complete w.r.t. the same evaluative semantics, though this is of no central interest here.

2. The natural deduction system

A BHK-like justification:
– A proof of $All\ X\ are\ Y$ is a (construction for) a *function* mapping a proof of $J\ is\ an\ X$ to a proof of $J\ is\ a\ Y$.

This is different from, though related to, the function involved in the *BHK*-justification for $\forall x.\phi(x)$, mapping an object o to a proof of $\phi(o)$.

– A proof of $Some\ X\ are\ Y$ is a pair of proofs of $J\ is\ an\ X$ and $J\ is\ a\ Y$.

This is reminiscent to the *BHK*-justification of *conjunction*, also constituting a pair of proofs.

The natural deduction rules

There is an *introduction-rule* and *elimination-rule* for each kind of propositions, presented below, together with proof-terms, to which we return later. The presentation is in Gentzen-style *ND*, using sequents.

$$S : u \vdash S : u \ \ (Ax) \quad \text{any } S \in SYL$$

$$\frac{\Gamma, [J\ is\ an\ X]_i : u \vdash J\ is\ a\ Y\ : M}{\Gamma \vdash All\ X\ are\ Y : \lambda u.M}\ (All - I_i)$$

$$\frac{\Gamma_1 \vdash All\ X\ are\ Y : M \quad \Gamma_2 \vdash J\ is\ an\ X : N}{\Gamma_1 \Gamma_2 \vdash J\ is\ a\ Y : (MN)}\ (All - E)$$

$$\frac{\Gamma_1 \vdash J\ is\ an\ X : M_1 \quad \Gamma_2 \vdash J\ is\ a\ Y : M_2}{\Gamma_1 \Gamma_2 \vdash Some\ X\ are\ Y : \langle M_1, M_2 \rangle}\ (Some - I)$$

$$\frac{\Gamma_1 \vdash Some\ X\ are\ Y : M \quad \Gamma_2, [J\ is\ an\ X]_i : u, [J\ is\ a\ Y]_i : v \vdash S : N}{\Gamma_1 \Gamma_2 \vdash S :\ \mathbf{let}\ \langle u, v \rangle = M\ \mathbf{in}\ N}\ (Some - E_i)$$

$$\frac{\Gamma, [\mathit{Some}\ X\ \mathit{are}\ Y]_i \vdash S}{\Gamma \vdash No\ X\ are\ Y}\ (No - I)_i^S$$

Here S is a *parameter proposition* not occurring in $\Gamma \cup \{Some\ X\ are\ Y\}$.

$$\frac{\Gamma_1 \vdash \mathit{Some}\ X\ \mathit{are}\ Y \quad \Gamma_2 \vdash\ NoXY}{\Gamma_1 \Gamma_2 \vdash S}\ (No - E)$$

We denote by $\vdash_{ND-Syl}$ derivability/provability in this system. As an example of a derivation using those rules, consider $Some\ X\ are\ Y,\ All\ Y\ are\ Z \vdash_{ND-syl^+} Some\ X\ are\ Z$.

$$\dfrac{Some\ X\ are\ Y : x \quad \dfrac{[J\ is\ an\ X]_i : u \quad \dfrac{[J\ is\ a\ Y]_i : v \quad All\ Y\ are\ Z : w}{J\ is\ a\ Z : (wv)}\ (All-E)}{Some\ X\ are\ Z : \langle u, (wv)\rangle}\ (Some-I)}{Some\ X\ are\ Z :\ \mathbf{let}\ \langle u, v\rangle\ = x\ \mathbf{in}\ \langle u, (wv)\rangle}\ (Some-E_i)$$

3. Properties of the Positive Fragment

3.1. Curry-Howard correspondence

We point out several observations about derivations in $ND-syl^+$.

1. The conclusion of an instance of application of the $(All-I)$ rule cannot serve as a premiss of another instance of application of the same rule.
2. The conclusion of an instance of application of the $(All-E)$ rule cannot serve as a major premiss of another instance of application of the same rule.
3. The conclusion of an instance of application of the rule $(Some-I)$ cannot serve as a premiss of another instance of application of the same rule.
4. The conclusion of an instance of application of the rule $(All-I)$ cannot serve as a premiss of an instance of application of the $(Some-I)$ rule.

Two important remarks about *discharge* of assumptions by the $(All-I)$-rule:

No vacuous discharge: The rule $(All-I)$ should not allow *vacuous discharge*; otherwise, the following unwarranted[1] derivation becomes possible.

$$\dfrac{J\ is\ a\ Y : u \vdash J\ is\ a\ Y : u}{J\ is\ a\ Y : u \vdash All\ X\ are\ Y : \lambda v.u}\ (All-I_{vac})$$

No multiple discharge: In the absence of the *Weakening* structural rule, *multiple discharge* becomes actually impossible, because there is no way to generate sequents, of the form, say, $\Gamma, J\ is\ an\ X,\ J\ is\ an\ X \vdash J\ is\ a\ Y$.

However, *contraction needs to be admitted*. To see the need for it, consider the following: $J\ is\ an\ X,\ All\ X\ are\ Y,\ All\ X\ are\ Z \vdash Some\ Y\ are\ Z$.

[1] Semantically, unsound ...

The assumption $J\ is\ an\ X$ has to be used twice, to eliminate both occurrences of All.

$$\dfrac{\dfrac{J\ is\ an\ X : x \quad All\ X\ are\ Y : y}{J\ is\ a\ Y : (yx)}\ (All-E) \quad \dfrac{J\ is\ an\ X : x \quad All\ X\ are\ Z : z}{J\ is\ a\ Z : (zx)}\ (All-E)}{Some\ Y\ are\ Z : \langle (yx), (zx)\rangle}\ (Some-I)$$

Note the $(Some - E)$ elimination rule, that *does not* allow *projection*. Indeed, we do not want $J\ is\ an\ X$ to be derivable from $Some\ X\ are\ Y$. The reduction-rule to be shown for harmony requires a *joint discharge* of its "J-assumptions". Thus, the resulting proof-terms are "almost" *linear*. This gives rise to the definition of Λ^{fl}, the subset of the set Λ of all λ-terms, referred to as *flat terms*.

Definition:(flat terms) Λ^{fl} is the smallest subset of Λ satisfying:

1. If u is a term-variable then $u \in \Lambda^{fl}$.
2. If $M, N \in \Lambda^{fl}$ and M is a variable or an abstraction-term, then $(MN) \in \Lambda^{fl}$.
3. If u is a term-variable and $M \in \Lambda^{fl}$ s.t. M ia a variable, or an application-term containing *exactly one* free occurrence of u, then $\lambda u.M \in \Lambda^{fl}$.
4. If $M_1, M_2 \in \Lambda^{fl}$, and none is of the form $\lambda x.N$, nor of the form $\langle P, Q\rangle$, then $\langle M_1, M_2\rangle \in \Lambda^{fl}$.
5. If $M, N \in \Lambda^{fl}$, $M \equiv \langle M_1, M_2\rangle$ or $M \equiv x$, and N is a pair-term or a **let**-term, then $\mathbf{let}\ \langle u, v\rangle = M\ \mathbf{in}\ N \in \Lambda^{fl}$.

While the $ND - syl^+$ calculus uses the the flat terms as its proof-terms, a subset of the (implicational fragment of the) Intuitionistic linear propositional calculus proof terms, it constitutes a completely different *typing* system for those terms. However, it enjoys similar properties to the latter, expressed in the following two theorems.

Theorem (flatness): If $\vdash_{ND-syl^+} S : M$, then $M \in \Lambda^{fl}$ and $free(M) = Subjects(\Gamma)$.

Theorem (subject construction): If $M \in \Lambda^{fl}$, then there exists a SYL^+ proposition S s.t. there exists a derivation $\mathcal{D}$ of $\Gamma \vdash_{ND-Syl^+} S : M$, where:

1. If $M \equiv u$ (a term-variable), then $\Gamma = S$ for some type S (a proposition in the Syllogistic fragment!), and $\mathcal{D}$ is the axiom $S : u \vdash_{ND-Syl^+} S : u$.
2. If $M \equiv (PQ)$, then the last step in $\mathcal{D}$ must be

$$\dfrac{\Gamma_1 \vdash All\ X\ are\ Y : P \quad \Gamma_2 \vdash J\ is\ a\ Y : Q}{\Gamma_1\Gamma_2 \vdash J\ is\ a\ Y : (PQ)}\ (All - E)$$

   for some partition $\Gamma = \Gamma_1\Gamma_2$.

3. If $M \equiv \lambda u.N$, then the last step in $\mathcal{D}$ must be

$$\dfrac{\Gamma, [J\ is\ an\ X]_i : u \vdash J\ is\ a\ Y\ : N}{\Gamma \vdash All\ X\ are\ Y : \lambda u.N}\ (All - I_i)$$

4. If $M \equiv \langle M_1, M_2 \rangle$, then the last step in $\mathcal{D}$ must be

$$\frac{\Gamma_1 \vdash J\ is\ an\ X : M_1 \quad \Gamma_2 \vdash J\ is\ a\ Y : M_2}{\Gamma_1 \Gamma_2 \vdash Some\ X\ are\ Y : \langle M_1, M_2 \rangle}\ (Some - I)$$

for some partition $\Gamma = \Gamma_1 \Gamma_2$.

5. If $M \equiv \mathbf{let}\ \langle u, v \rangle = P\ \mathbf{in}\ N$, then the last step in $\mathcal{D}$ must be

$$\frac{\Gamma_1 \vdash Some\ X\ are\ Y : P \quad \Gamma_2, [J\ is\ an\ X]_i : u, [J\ is\ a\ Y]_i : v \vdash S : N}{\Gamma_1 \Gamma_2 \vdash S : \mathbf{let} \langle u, v \rangle = P\ \mathbf{in}\ N}\ (Some - E_i)$$

3.2. A correspondence with a sub-Intuitionistic fragment

Based on the identity of proof-terms, there is a natural isomorphism between SYL^+ and a fragment $ILprop$ of the implicational fragment of the Intuitionistic linear propositional calculus. Denote by $\vdash_{i-lint}$ the derivability in the standard natural-deduction proof-system for the latter (e.g., Negri 2002). Let the propositional variables in $ILprop$ be in 1-1 correspondence with the predicate variables in SYL. For simplicity, we just identify both sets. Define a syntactic mapping $\Pi : SYL^+ \Longrightarrow ILprop$ by: $\Pi(J\ is\ an\ X) = X$, $\Pi(All\ X\ are\ Y) = X \multimap Y$, $\Pi(Some\ X\ are\ Y) = X \bullet Y$ Obviously, if ϕ is in the range of Π, ϕ *has no nested implications*; also, there are no directly nested occurrences of pairing. Furthermore, abstraction-terms cannot be paired. Hence the name 'flat'. Extending Π naturally to sets Γ, we get as a conclusion from sharing proof-terms that $\Gamma \vdash_{ND-syl^+} S : M \iff \Pi(\Gamma) \vdash_{i-lint} \Pi(\phi) : M$ (where corresponding subject variables are assumed in Γ and $\Pi(\Gamma)$).

A semantic digression:

The only tautologies in $ILprop$ are of the form $X \multimap X$, reflecting the fact that the only validities in SYL^+ are of the form $All\ X\ are\ X$ (cf. Moss 2005).

3.3. Harmony

We now show that $ND - syl^+$ satisfies *harmony*, as expressed via the local soundness and completeness, (Pfenning and Davies 2001) providing *reduction* and *expansion* steps, embodying Prawitz's *inversion principle* Prawitz 1965; Prawitz 1971. For better readability, we employ the Prawitz style presentation of natural deduction.

$All\ X\ are\ Y$ **– local soundness:**

$$\frac{\dfrac{\begin{matrix}[J\ is\ a\ X]_i \\ \mathcal{D}_1 \\ J\ is\ a\ Y\end{matrix}}{All\ X\ are\ Y}\ (All - I_i) \quad \begin{matrix}\mathcal{D}_2 \\ J\ is\ a\ X\end{matrix}}{J\ is\ a\ Y}\ (All - E) \quad \rightsquigarrow_r \quad \begin{matrix}\mathcal{D}_2 \\ J\ is\ a\ X \\ \mathcal{D}_1 \\ J\ is\ a\ Y\end{matrix}$$

$All\ X\ are\ Y$ **– local completeness:**

$$\begin{matrix}\mathcal{D} \\ All\ X\ are\ Y\end{matrix} \quad \rightsquigarrow_e \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D} \\ All\ X\ are\ Y\end{matrix} \quad [J\ is\ a\ X]_i}{J\ is\ a\ Y}\ (All\ E)}{All\ X\ are\ Y}\ (All\ I)_i$$

$Some\ X\ are\ Y$ **– local soundness:**

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ J\ is\ an\ X & J\ is\ a\ Y\end{matrix}}{Some\ X\ are\ Y}(Some-I) \qquad \begin{matrix}[J\ is\ an\ X]_i\ \ [J\ is\ a\ Y]_i\\ \mathcal{D}_3\\ S\end{matrix}}{S}(Some-E)_i \quad \rightsquigarrow_R \quad \begin{matrix}\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ J\ is\ an\ X & J\ is\ a\ Y\end{matrix}\\ \mathcal{D}_3\\ S\end{matrix}$$

$Some\ X\ are\ Y$ **– local completeness:**

$$\begin{matrix}\mathcal{D}\\ Some\ X\ are\ Y\end{matrix} \quad \rightsquigarrow_E \quad \dfrac{\begin{matrix}\mathcal{D}\\ Some\ X\ are\ Y\end{matrix} \qquad \dfrac{[J\ is\ an\ X]_i \quad [J\ is\ a\ Y]_i}{Some\ X\ are\ Y}(Some-I)}{Some\ X\ are\ Y}(Some-E_i)$$

3.4. Decidability of Provability

Strictly speaking, $ND - syl^+$ does not enjoy the sub-formula property, simply because propositions in SYL^+ (and generally in SYL) do not have sub-formulas. However, both of X, Y are sub-formulas of $X \multimap Y = \Pi(All\ X\ are\ Y)$, and $\vdash_{i-lint}$ does enjoy the sub-formula property.

Thus, a straightforward way to decide $\Gamma \vdash_{ND-syl^+} \phi$ is to decide $\Pi(\Gamma) \vdash_{i-lint} \Pi(\phi)$, using the known algorithm based on the sub-formula property of $\vdash_{i-lint}$. Obviously, a *direct* decision algorithm can be obtained too.

4. Conclusions

Clearly, the calculus presented here, in its preliminary for, constitutes only a modest first step toward the goal of PTS for NL. The real challenge, even for this small fragment, is the incorporation into a grammar, devising a *lexicalized* PTS. This is currently under investigation.

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